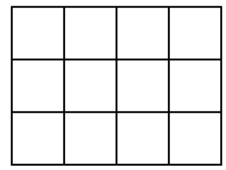
1. How many diagonals are there in a regular polygon of n sides?

2. How many 5−digit numbers can be formed using the digits $1, 2, 3, 5, 7, 8$, and 9, if

   a) repetition of digits is allowed   b) repetition of digits is not allowed?

3. Prove that for every positive number x, $x + \dfrac{1}{x} \geq 2$.

4. Use systematic listing to answer the following questions.
   a) How many rectangles are there on the picture?
   b) How many squares are there on the picture?



5. a) Find all real numbers with the following property: the number is 2 greater than its own reciprocal.

   b) Prove that the number(s) you found has that property.

6. Compute the exact value of each of the following. Simplify your answer.

   a) $\sin 60° - \tan 30°$ b) $\sec 45° - \sin 45°$ c) $\tan 30° \cdot \tan 45° \cdot \tan 60°$ d) $\sin^2 30° + \cos^2 30°$

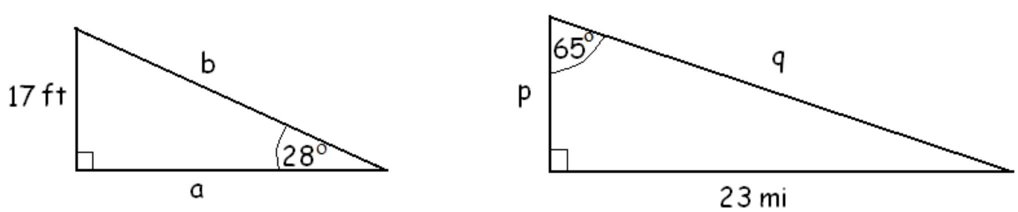
7. A right triangle has sides $5\,\mathrm{ft}$, $12\,\mathrm{ft}$, and $13\,\mathrm{ft}$ long.

   a) State the value of all six trigonometric functions of α if α is the angle opposite the $5\,\mathrm{ft}$ long side.

   b) State the value of all six trigonometric functions of β if β is the angle opposite the $12\,\mathrm{ft}$ long side.

   c) Compute the approximate value (up to four or more decimal places) of the measure of the smallest angle in the triangle.

8. Compute the exact value and the approximate value for each of a, b, p, and q, based on the picture below.



9. The hypotenuse of a right triangle is 74 units. The difference between the lengths of the other two sides is 46 units.

   a) How long are the sides of this triangle?

   b) Use your calculator to find an approximate value of the smallest angle in the triangle. Present your answer in degrees, accurate up to four or more decimals.

10. Consider a circle of radius 12 units and with a center C. P is a point located 30 units away from C. We draw a tangent line from P to the circle. The point of tangency is Q.

    a) Compute the exact value of $d(P, Q)$ (that is the distance between P and Q).

    b) Let α denote the angle CPQ. Compute the exact value of all six trigonometric functions of α.

11. Find the exact value and an approximate value for the angle that is formed between the line $y = \dfrac{1}{2}x$ and the positive part of the x−axis.

12. Compute the perimeter and area of the 10-sided regular polygon that is written into a circle with radius $3\,\mathrm{cm}$. Present both exact and approximate values for the answers.

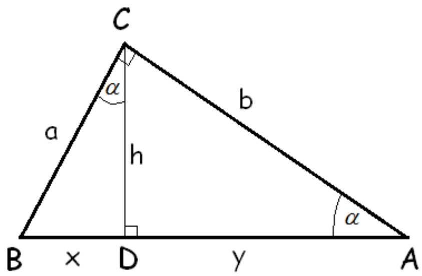
13. Solve each of the following inequalities.

a) $6x + x^2 \leq 16$
b) $6x + x^2 \leq 15$
c) $x^2 > 4x$
d) $x^2 > 4x + 1$
e) $16x^2 \leq 8x - 1$
f) $(2x-1)^2 \leq 3(x-1)^2$
g) $x^2 - 5x + 14 \geq x + 3$

14. Consider the right triangle shown on the picture.

Prove each of the following.

a) $h = \sqrt{xy}$
b) $b = \sqrt{y(x+y)}$



15. A company finds that if it prices its product at \$50, then it can sell 800 items. For every dollar increase in the price, the company will sell 4 less items.

a) What is the maximum revenue possible, and what price guarantees that maximal revenue?
b) What price range will guarantee a revenue greater than \$50 400?

16. Suppose that m and n are real number such that m is 10 less than three times n. Find each of the following.

a) the smallest value of $n^2 + m^2$
b) the smallest value of nm
c) the greatest value of $n^2 - m^2$

17. Suppose that m and n are real number such that m is 10 less than three times n. For what values of n will the value of $m^2 + n^2$ be greater than 260?

18. Solve each of the given systems of equations.

a) $\begin{cases} (x-1)^2 + (y+3)^2 = 10 \\ 3y = -x - 8 \end{cases}$

b) $\begin{cases} (x+8)^2 + (y-2)^2 = 50 \\ y + 7x = -4 \end{cases}$

c) $\begin{cases} (x-4)^2 + (y+6)^2 = 20 \\ x + 2y = 7 \end{cases}$

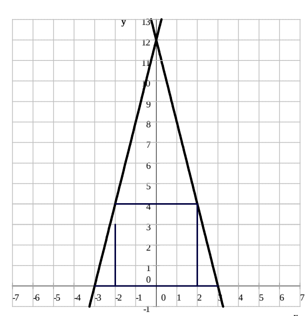
d) $\begin{cases} (x+8)^2 + (y-2)^2 = 50 \\ 7y = x + 22 \end{cases}$

19. The hypotenuse of a right triangle is three feet shorter than four times its shortest side. The other side is three feet longer than three times its shortest side.

a) Find the sides of the triangle.
b) Find an approximate value of the smallest angle in the triangle. Present your answer accurate up to three or more decimal places.

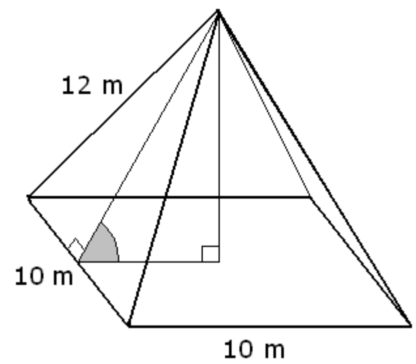
20. Prove that for any acute angle α, $\sin\alpha + \cos\alpha > 1$.

21. Let l_1 and l_2 denote the lines $y = 4x + 12$ and $y = -4x + 12$, respectively. Let R be a rectangle with vertical and horizontal sides, where one horizontal side is on the $x-$axis and the vertices connecting the other horizontal side lie on the lines l_1 and l_2, above the $x-$axis. What is the maximal value of the area of such a rectangle?



22. a) Find the distance between the points $A(3,-7)$ and $B(-2,5)$.
    b) Find the distance between the points $P(x,y)$ and $Q(3,8)$.

23. Consider a square based straight pyramid as shown on the picture. The base is a square with sides $10\,\mathrm{m}$ long, and all other edges are $12\,\mathrm{m}$ long. Find the exact and approximate value of the angle that is formed between a triangular face and the square base. The angle is marked on the picture.



Answers

1. $\dfrac{n(n-3)}{2}$ 2. a) $7^5 = 16\,807$ b) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

3. Proof: Let x be any positive number. Then $(x-1)^2 \geq 0$ since it is a square.

$$\begin{aligned} (x-1)^2 &\geq 0 \\ x^2 - 2x + 1 &\geq 0 && \text{add } 2x \\ x^2 + 1 &\geq 2x && \text{divide by } x;\ \text{recall that } x > 0 \\ x + \frac{1}{x} &\geq 2 \end{aligned}$$

4. a) 60 b) 20

5. a) $1-\sqrt{2}$ and $1+\sqrt{2}$
   b) If $x = 1-\sqrt{2}$, then its reciprocal is

$$\frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}} \cdot \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} = -1-\sqrt{2}$$

If we add 2 to this number, we get $-1-\sqrt{2}+2 = 1-\sqrt{2}$, and so x is indeed 2 greater than its reciprocal. And if $x = 1+\sqrt{2}$, then its reciprocal is

$$\frac{1}{1+\sqrt{2}} = \frac{1}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1$$

which is indeed 2 less than $\sqrt{2}+1$.

6. a) $\dfrac{\sqrt{3}}{6}$ b) $\dfrac{\sqrt{2}}{2}$ c) 1 d) 1

7. a) $\sin\alpha = \dfrac{5}{13}$ $\cos\alpha = \dfrac{12}{13}$ $\tan\alpha = \dfrac{5}{12}$ $\csc\alpha = \dfrac{13}{5}$ $\sec\alpha = \dfrac{13}{12}$ $\cot\alpha = \dfrac{12}{5}$
   b) $\sin\beta = \dfrac{12}{13}$ $\cos\beta = \dfrac{5}{13}$ $\tan\beta = \dfrac{12}{5}$ $\csc\beta = \dfrac{13}{12}$ $\sec\beta = \dfrac{13}{5}$ $\cot\beta = \dfrac{5}{12}$ c) $22.619\,865^\circ$

8. $a = \dfrac{17}{\tan 28^\circ}\,\mathrm{ft} \approx 31.9724\,\mathrm{ft}$ $b = \dfrac{17}{\sin 28^\circ}\,\mathrm{ft} \approx 36.21093\,\mathrm{ft}$
   $p = \dfrac{23}{\tan 65^\circ}\,\mathrm{mi} \approx 10.72507614\,\mathrm{mi}$ $q = \dfrac{23}{\sin 65^\circ}\,\mathrm{mi} \approx 25.377\,692\,\mathrm{mi}$

9. a) 24, 70, and 74 units long b) $18.924\,644^\circ$

10. a) $\sqrt{756} = 6\sqrt{21}$ b) $\sin\alpha = \dfrac{2}{5}$ $\cos\alpha = \dfrac{\sqrt{21}}{5}$ $\tan\alpha = \dfrac{2}{\sqrt{21}}$ $\csc\alpha = \dfrac{5}{2}$ $\sec\alpha = \dfrac{5}{\sqrt{21}}$ $\cot\alpha = \dfrac{\sqrt{21}}{2}$

11. exact value: $\tan^{-1}\left(\dfrac{1}{2}\right)$ approximation: $26.565\,051\,18^\circ$

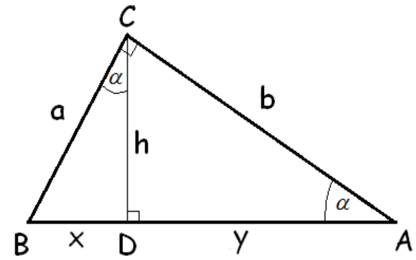
12. $P = 60\sin 18^\circ\text{ cm} \approx 18.541\,02\text{ cm}$ $A = 90\sin 18^\circ\cos 18^\circ\text{ cm}^2 \approx 26.450\,336\text{ cm}^2$

13. a) $[-8, 2]$ b) $\left[-3 - 2\sqrt{6}, -3 + 2\sqrt{6}\right]$ c) $(-\infty, 0) \cup (4, \infty)$ d) $\left(-\infty, 2 - \sqrt{5}\right) \cup \left(2 + \sqrt{5}, \infty\right)$ e) $\left\{\dfrac{1}{4}\right\}$
f) $-1 - \sqrt{3} \le x \le -1 + \sqrt{3}$ g) $\mathbb{R}$

14. a) $h = \sqrt{xy}$
Proof: Triangles BDC and CDA are similar. Thus

$$\frac{\text{side opposite } \alpha}{\text{side opposite } \beta} = \frac{x}{h} = \frac{h}{y}$$

$$\frac{x}{h} = \frac{h}{y} \iff h^2 = xy \iff h = \sqrt{xy}$$



b) $b = \sqrt{y(x+y)}$
Proof: Triangles ACB and ADC are similar. Thus

$$\frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{y}{b} = \frac{b}{x+y}$$

$$\frac{y}{b} = \frac{b}{x+y} \iff b^2 = y(x+y) \iff b = \sqrt{y(x+y)}$$

15. a) max revenue is \$62 500, if the price is \$125 b) between \$70 and \$180 16. a) 10 b) $-\dfrac{25}{3}$ c) $\dfrac{25}{2}$

17. interval notation: $(-\infty, -2) \cup (8, \infty)$ inequality notation: $n < -2$ or $n > 8$

18. a) $(-2, -2)$ and $(4, -4)$ b) $(-1, 3)$ c) no solution d) $(-1, 3)$ and $(-15, 1)$

19. a) 7 ft, 24 ft, and 25 ft b) $\sin^{-1}\left(\dfrac{7}{25}\right) \approx 16.260\,2^\circ$

20. The triangle inequality states that $a + b > c$. Divide both sides by c. $c > 0$ 21. 18

22. a) 13 unit b) $\sqrt{(x-3)^2 + (y-8)^2}$ 23. exact value: $\cos^{-1}\left(\dfrac{5}{\sqrt{119}}\right)$ approximation: $62.719\,361\,28^\circ$