Sample Problems

1. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

2. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

3. An open box (no top) with a square base is to have a volume of $60\,\mathrm{in}^3$. What dimensions would guarantee the least amount of material needed?

4. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

5. We are designing a poster to contain $60\,\mathrm{in}^2$ of printing with a $2-$ inch wide margin at the top and bottom and a $1-$ inch wide margin at each side. What overall dimensions will minimize the amount of paper used?

6. Consider all lines with negative slopes that pass through the point $P(8,2)$. Let us denote the origin by O, the $x-$intercept of the line by A and its $y-$intercept by B. What is the smallest possible area of triangle OAB?

Practice Problems

1. We are designing a poster to contain $60\,\mathrm{in}^2$ of printing with a $3-$ inch wide margin at the top and bottom and a $2-$ inch wide margin at each side. What overall dimensions will minimize the amount of paper used?

2. Let x and y be positive numbers such that $xy = 1$. Find the lowest possible value of $x^3 + 2y^3$.

3. We would like to constract an open box with a square base. The box to have a volume of $200\,\mathrm{in}^3$. What dimensions would guarantee that the box can be made using the least amount of material?

4. A company wants to manufacture cylindrical aluminum cans with a volume of 200π cubic centimeters. What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

5. Consider all lines with negative slopes that pass through the point $P(3,12)$. Let us denote the origin by O, the $x-$intercept of the line by A and its $y-$intercept by B. What is the smallest possible area of triangle OAB?

6. We would like to design a flowerbed in the shape of a circular sector. If the area of the sector needs to be $20\,\mathrm{m}^2$, then what is the smallest possible perimeter?

Sample Problems - Answers

1. 40
2. no,the lowest possible cost is $\$300$ when the box is to be $5\,\mathrm{m}$ by $5\,\mathrm{m}$ by $10\,\mathrm{m}$
3. $x = \sqrt[3]{120}$ and $y = \dfrac{1}{2}\sqrt[3]{120}$
4. $r = \sqrt[3]{\dfrac{500}{\pi}} \simeq 5.419\,26$ and $h = 2\sqrt[3]{\dfrac{500}{\pi}} = 2r \simeq 10.838\,521\,402\,785\,8$
5. horizontal side $\sqrt{30} + 2$ inches long, and the vertical side $2\sqrt{30} + 4$ inches long
6. $m = -\dfrac{1}{4}$

Practice Problems - Answers

1. the print area is $2\sqrt{10}\,\mathrm{in}$ (horizontal) and $3\sqrt{10}\,\mathrm{in}$ (vertical),
   the paper is $2\sqrt{10} + 4\,\mathrm{in}$ (horizontal) and $3\sqrt{10} + 6\,\mathrm{in}$ (vertical)
2. $2\sqrt{2}$ (when $x = \sqrt[6]{2}$)
3. base: $\sqrt[3]{400}\,\mathrm{in}$ by $\sqrt[3]{400}\,\mathrm{in}$ height: $\dfrac{1}{2}\sqrt[3]{400}\,\mathrm{in}$
4. $r = \sqrt[3]{100}\,\mathrm{cm}$ and $h = 2\sqrt[3]{100}\,\mathrm{cm}$
5. $m = -4$
6. $r = 2\sqrt{5}$ $\alpha = 2$rad

Sample Problems - Solutions

1. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

Solution: We solve for a in terms of b : $a = \dfrac{10}{b}$. Then the expression $a^2 + 4b^2$ becomes

$$P(b) = \left(\frac{10}{b}\right)^2 + 4b^2 = 4b^2 + \frac{100}{b^2} = 4b^2 + 100b^{-2}$$

We differentiate this:

$$\begin{aligned} P'(b) &= 8b + 100(-2)b^{-3} = 8b - \frac{200}{b^3} = \frac{8b^4 - 200}{b^3} \\ &= \frac{8(b^4 - 25)}{b^3} = \frac{8(b^2+5)(b^2-5)}{b^3} = \frac{8(b^2+5)(b+\sqrt{5})(b-\sqrt{5})}{b^3} \end{aligned}$$

The critical numbers for P are $-\sqrt{5}$, 0, and $\sqrt{5}$. All relative maximums or minimums will be here. We can figure out when P' is positive and negative by sorting out the signs of each factor in the numerator and denominator.

	$b < -\sqrt{5}$	$-\sqrt{5} < b < 0$	$0 < b < \sqrt{5}$	$b > \sqrt{5}$
(b^2+5)	$+$	$+$	$+$	$+$
$(b+\sqrt{5})$	$-$	$+$	$+$	$+$
$(b-\sqrt{5})$	$-$	$-$	$-$	$+$
b^3	$-$	$-$	$+$	$+$
P'	$-$	$+$	$-$	$+$

Based on the signs of P' only, P has a relative minimum at $b = -\sqrt{5}$ and $\sqrt{5}$ and a relative maximum at 0. However, the function does not have a relative maximum at zero. Looking at the formula for the original function, $P(b) = 4b^2 + \dfrac{100}{b^2}$, we see that there is a vertical asymptote and the graph shoots up toward plus infinity on both sides of the asymptote. Not to mention tha fact that a and b must both be positive. Since b must be positive, we may consider P on the domain $(0, \infty)$. On this domain, P is continuous and differentiable everywhere, is decreasing on $(0, \sqrt{5})$ and increasing on $(\sqrt{5}, \infty)$ and so P has an absolute minimum at $b = \sqrt{5}$.

If $b = \sqrt{5}$, then

$$P\left(\sqrt{5}\right) = \left(\frac{10}{\sqrt{5}}\right)^2 + 4\left(\sqrt{5}\right)^2 = \frac{100}{5} + 4 \cdot 5 = 20 + 20 = 40$$

Thus the smallest possible value of $a^2 + 4b^2$ is $\boxed{40}$.

2. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

   Solution: Let x denote the side of the square base, and h denote the height of the box. Then $V = hx^2$ gives us

$$\begin{aligned} hx^2 &= 250 \\ h &= \frac{250}{x^2} \end{aligned}$$

   We now set up the cost function, $C(x)$. The top and bottom each cost \$2 per square meter, and have area x^2. The four sides each have area $xh = x\left(\frac{250}{x^2}\right) = \frac{250}{x}$ and cost \$1 per square meter. Thus

$$C(x) = 2 \cdot 2 \cdot x^2 + 4 \cdot 1 \cdot \frac{250}{x} = 4x^2 + \frac{1000}{x} = 4x^2 + 1000x^{-1}$$

   We are looking for the maximum of $C(x)$. We will differentiate C first.

$$\begin{aligned} C'(x) &= 8x + 1000(-1)x^{-2} = 8x - \frac{1000}{x^2} = \frac{8x^3 - 1000}{x^2} = \frac{8(x^3 - 125)}{x^2} \\ &= \frac{8(x-5)(x^2 + 5x + 25)}{x^2} \end{aligned}$$

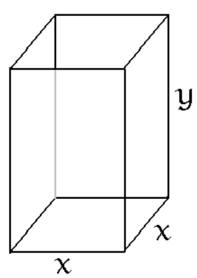
   The last form shows that C' has only one zero, at $x = 5$. Since both x^2 and $x^2 + 5x + 25$ are positive for all values of x, C' will change sign from negative to positive at $x = 5$, indicating a minimum of C. Thus, the lowest possible cost will be associated with $x = 5$. The actual cost is then

$$C(5) = 4 \cdot 5^2 + \frac{1000}{5} = 300$$

   Thus, we can not construct this box for less than \$300.

3. An open box (no top) with a square base is to have a volume of $60\,\text{in}^3$. What dimensions would guarantee the least amount of material needed?

   Solution 1: Let us denote the sides of the square base by x and the vertical side by y.



   The volume of the box is $V = x^2y$, so we have that

$$60 = x^2 y \qquad \text{solve for } y\text{:} \quad y = \frac{60}{x^2}$$

   The amount of material needed: x^2 for the bottom and $4xy$ for the vertical sides. So the function, whose minimum we are to find, is

$$A = x^2 + 4xy = x^2 + 4x\left(\frac{60}{x^2}\right) = x^2 + \frac{240}{x} = x^2 + 240x^{-1}$$

   We differentiate: $\frac{d}{dx}\left(x^2 + \frac{240}{x}\right) = 2x - \frac{240}{x^2} = \frac{2x^3 - 240}{x^2}$

   Recall the difference of cubes theorem:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

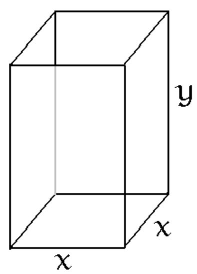
where the second factor is a sum of two squares, always positive. (same as $\left(A+\frac{B}{2}\right)^2+\frac{3}{4}B^2$). Using this, we can factor the denominator:

$$f'(x)=\frac{2x^3-240}{x^2}=\frac{2\left(x^3-120\right)}{x^2}=\frac{2\left(x-\sqrt[3]{120}\right)\left(x^2+\sqrt[3]{120}x+\left(\sqrt[3]{120}\right)^2\right)}{x^2}$$

The denominator is always positive, and so is the second, longer factor. The only factor that changes sign is the line $y=x-\sqrt[3]{120}$. At $x=\sqrt[3]{120}$, this line changes sign from negative to positive, and so does f'. Therefore, f has a relative minimum at $x=\sqrt[3]{120}$.

Recall that $y=\frac{60}{x^2}$. So $y=\frac{60}{\left(\sqrt[3]{120}\right)^2}=\frac{60}{\left(\sqrt[3]{120}\right)^2}\cdot\frac{\sqrt[3]{120}}{\sqrt[3]{120}}=\frac{60\sqrt[3]{120}}{\left(\sqrt[3]{120}\right)^3}=\frac{60\sqrt[3]{120}}{120}=\frac{\sqrt[3]{120}}{2}=\frac{x}{2}$. So the dimensions that need the least amount of material are: $\boxed{x=\sqrt[3]{120}\text{ and }y=\frac{1}{2}\sqrt[3]{120}}$.

Solution 2. Let us denote the sides of the square base by x and the vertical side by y.



The volume of the box is $V=x^2y$, so we have that

$$60=x^2y \qquad \text{solve for } y: \quad y=\frac{60}{x^2}$$

The amount of material needed: x^2 for the bottom and $4xy$ for the vertical sides.
So the function, whose minimum we are to find, is

$$A=x^2+4x=x^2+4x\left(\frac{60}{x^2}\right)=x^2+\frac{240}{x} \qquad f(x)=x^2+\frac{240}{x}$$

We differentiate: $f'(x)=2x-\frac{240}{x^2}$ and again: $f''(x)=2+\frac{480}{x^3}$

Find the critical numbers: Solve $f'(x)=0$ for x.

$$\begin{aligned} f'(x) &= 0\\ 2x-\frac{240}{x^2} &= 0\\ 2x &= \frac{240}{x^2}\\ 2x^3 &= 240\\ x^3 &= 120\\ x &= \sqrt[3]{120}\end{aligned}$$

Since x is a side of the box, it must be a positive number. If so, the second derivative

$$f''(x)=2+\frac{480}{x^3}$$

is positive for all values of x in the domain, including at $x=\sqrt[3]{120}$. By the second derivative test, f has a minimum at $x=\sqrt[3]{120}$. Recall that $y=\frac{60}{x^2}$. So $y=\frac{60}{\left(\sqrt[3]{120}\right)^2}=\frac{60}{\left(\sqrt[3]{120}\right)^2}\cdot\frac{\sqrt[3]{120}}{\sqrt[3]{120}}=\frac{60\sqrt[3]{120}}{\left(\sqrt[3]{120}\right)^3}=\frac{60\sqrt[3]{120}}{120}=\frac{\sqrt[3]{120}}{2}=\frac{x}{2}$.

So the dimensions that need the least amount of material are: $\boxed{x=\sqrt[3]{120}\text{ and }y=\frac{1}{2}\sqrt[3]{120}}$.

4. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?
Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$\pi r^2 h = 1000 \qquad h = \frac{1000}{\pi r^2}$$

The domain is $(0, \infty)$

$$\begin{aligned} S(r) &= 2\pi rh + 2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r} \\ S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} \end{aligned}$$

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r &= \frac{2000}{r^2} \\ \pi r^3 &= 500 \implies r = \sqrt[3]{\frac{500}{\pi}} \approx 5.41926 \end{aligned}$$

and

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi\left(500^{2/3}\right)\left(\pi^{-2/3}\right)} = \frac{2\cdot 500}{\left(500^{2/3}\right)\left(\pi^{1/3}\right)} = \frac{2\cdot 500^{1/3}}{\left(\pi^{1/3}\right)} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.83852$$

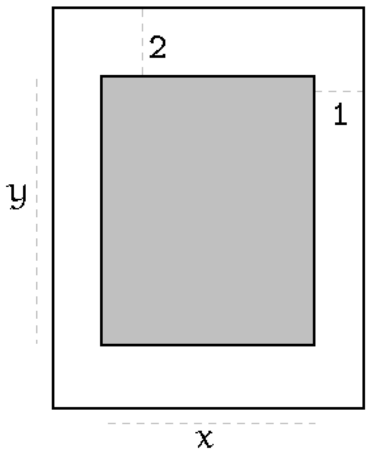
But is this an absolute minimum we found?

$$S''(r) = 4\pi + \frac{4000}{r^3}$$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero and positive after. This implies that S is decreasing before and increasing after, and so we indeed found the absolute minimum.

5. We are designing a poster to contain $60\,\text{in}^2$ of printing with a $2-$ inch wide margin at the top and bottom and a $1-$ inch wide margin at each side. What overall dimensions will minimize the amount of paper used?

Solution: Let x be the horizontal side of the printing and y the vertical side of the printing.



Then

$$xy = 60 \quad \Longrightarrow \quad y = \frac{60}{x}$$

The entire page then has sides $x+2$ and $y+4$ and therefore area $A = (x+2)(y+4)$. We are looking for the minimum value of this expression.

$$\begin{aligned} A &= (x+2)(y+4) = xy + 4x + 2y + 8 && \text{recall that } xy = 60 \\ &= 60 + 4x + 2y + 8 = 68 + 4x + 2y && \text{recall that } y = \frac{60}{x} \\ A(x) &= 68 + 4x + 2\left(\frac{60}{x}\right) = 68 + 4x + \frac{120}{x} \end{aligned}$$

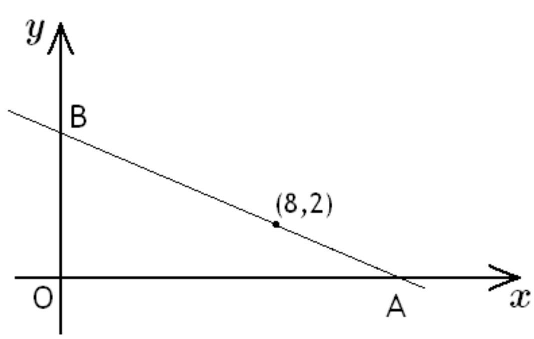
We differentiate $A(x)$

$$A'(x) = 4 - \frac{120}{x^2} = \frac{4x^2 - 120}{x^2} = \frac{4(x^2 - 30)}{x^2} = \frac{4(x + \sqrt{30})(x - \sqrt{30})}{x^2}$$

The denominator is always positive. The numerator is a quadratic expression, positive on $(-\infty, -\sqrt{30})$ and $(\sqrt{30}, \infty)$ and negative on $(-\sqrt{30}, \sqrt{30})$. At $x = \sqrt{30}$, the derivative A' changes sign from negative to positive, indicating a minimum. Therefore, $x = \sqrt{30}$, and $y = \frac{60}{x} = \frac{60}{\sqrt{30}} = \frac{60\sqrt{30}}{30} = 2\sqrt{30}$. The sheet of paper than needs to have sides $x + 2 = \sqrt{30} + 2$ and $y + 4 = 2\sqrt{30} + 4$. So the sides must be $\boxed{\sqrt{30} + 2 \text{ inches and } 2\sqrt{30} + 4 \text{ inches}}$ long.

6. Consider all lines with negative slopes that pass through the point $P(8,2)$. Let us denote the origin by O, the $x-$intercept of the line by A and its $y-$intercept by B. What is the smallest possible area of triangle OAB?

Solution: Let $m < 0$ be the slope of the line. Using the point-slope form, the equation of the line AB is $y - 2 = m(x-8)$ or $y = m(x-8) + 2$.



Let us find the intercepts in terms of m. If $x = 0$, then $y = m(x-8) + 2$ becomes $y = m(-8) + 2 = -8m + 2$. Thus the $y-$intercept is $(0, -8m+2)$.

If $y = 0$, then $y = m(x-8) + 2$ becomes $0 = m(x-8) + 2$ We solve for x.

$$\begin{aligned} 0 &= m(x-8) + 2 \\ -2 &= m(x-8) \\ -\frac{2}{m} &= x - 8 \\ 8 - \frac{2}{m} &= x \end{aligned}$$

Thus the $x-$intercept is $\left(8 - \frac{2}{m}, 0\right)$.

The area of triangle OAB is $A(m) = \frac{1}{2}(-8m+2)\left(8 - \frac{2}{m}\right) = -32m + 16 - \frac{2}{m}$. We differentiate $A(m)$

$$A'(m) = -32 + \frac{2}{m^2} \quad \text{we solve for the zeroes of } A'(m)$$

$$\begin{aligned} -32 + \frac{2}{m^2} &= 0 \\ \frac{2}{m^2} &= 32 \\ 2 &= 32m^2 \\ \frac{1}{16} &= m^2 \\ m &= \pm\frac{1}{4} \end{aligned}$$

Because $m < 0$, $m = -\frac{1}{4}$. We differentiate again:

$$A'(m) = -32 + \frac{2}{m^2} \quad \Longrightarrow \quad A''(m) = -\frac{4}{m^3}$$

Since m is negative, so is m^3. Therefore, $-\frac{4}{m^3}$ is positive for all $m < 0$, indicating a minimum at $m = -\frac{1}{4}$ by the second derivative test.

Another way to veify that A has a minimum at $x = -\frac{1}{4}$ is to look at $A'(m)$.

$$A'(m) = -32 + \frac{2}{m^2} = \frac{-32m^2 + 2}{m^2} = \frac{-32\left(m^2 - \frac{1}{16}\right)}{m^2} = \frac{-32\left(m + \frac{1}{4}\right)\left(m - \frac{1}{4}\right)}{m^2}$$

The denominator is always positive. The numerator is a downward opening parabola, negative on $\left(-\infty, -\frac{1}{4}\right)$ and on $\left(\frac{1}{4}, \infty\right)$, and positive on $\left(-\frac{1}{4}, \frac{1}{4}\right)$. Therefore, $A'(m)$ changes sign from negative to positive, indicating a minimum at $\boxed{m = -\frac{1}{4}}$.

