Students must be able to correctly state the following theorems: Intermediate Value Theorem (both forms), Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem, the second derivative test.

Students must be able to prove the following theorems:

- Differentiating functions using the definition (limit of the differential quotient)
- Prove that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(a^x) = a^x \cdot \ln a$, $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1}$
- If a function is differentiable at a number x, then it is continuous there.
- The product rule and quotient rule for derivatives, Rolle's Theorem, The Mean Value Theorem.
- (New!) If a sequence is increasing and bounded from above, then it has a limit.

Review Problems

1. Prove the quotient rule of dervivatives.
2. Compare the domains of $f(x) = \sin^{-1} x$ and of $f'(x)$. What is similar? What is different? Can you explain the difference?
3. Compute each of the given limits.

   a) $\lim_{x\to 0}\left(x^2\cos\left(\frac{1}{x}\right)\right)$

   b) $\lim_{x\to 0}\frac{6x-\sin 2x}{2x+3\sin 4x}$

   c) $\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{2+\frac{1}{n}}+\frac{1}{2+\frac{2}{n}}+\frac{1}{2+\frac{3}{n}}+\ldots+\frac{1}{2+\frac{n}{n}}\right)$

   d) $\lim_{n\to\infty}\frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\ldots+\sin(\pi)\right)$

4. Suppose that f and g are differentiable function such that $f(0) = 0$, $g(0) = 1$, $f'(x) = g(x)$, and $g'(x) = -f(x)$.
   a) Compute the exact value of $f^2(0) + g^2(0)$.
   b) Use differentation to prove that the function $h(x) = f^2(x) + g^2(x)$ is constant.
   c) What can we conclude?
5. Compute each of the given indefinite integrals.

   a) $\int e^{3x+1}dx$

   b) $\int \frac{8x-3}{2x-1}dx$

   c) $\int \frac{1}{x^2+1}dx$

   d) $\int \frac{1}{25x^2+1}dx$

   e) $\int \frac{x^2}{x^2+1}dx$

   f) $\int \sin 2x dx$

   g) $\int \sin^2 x dx$

   (Hint: $\cos 2x = 1 - 2\sin^2 x$)

   h) $\int \frac{1}{\sqrt{1-49x^2}}dx$

   i) $\int \frac{1}{(2x-5)^7}dx$

   j) $\int \sqrt{5x+1}dx$

6. Compute the exact value of each of the given definite integrals.

a) $\displaystyle\int_{-3}^{3} \left(12x^3 - x\right) dx$ d) $\displaystyle\int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$ g) $\displaystyle\int_{0}^{1/2} \frac{1}{\sqrt{1-4x^2}} dx$

b) $\displaystyle\int_{1}^{2} \frac{1}{x} dx$ e) $\displaystyle\int_{-1}^{1} \frac{1}{2}\left(e^x + e^{-x}\right) dx$ h) $\displaystyle\int_{0}^{3} 2^x dx$

c) $\displaystyle\int_{0}^{1} \frac{1}{1+x^2} dx$ f) $\displaystyle\int_{-\ln 2}^{\ln 2} e^x dx$ i) $\displaystyle\int_{0}^{3} \sqrt{5x+1} dx$

7. Differentiate each of the following.

a) $f(x) = e^{e^{e^x}}$ c) $f(x) = \displaystyle\int_{1}^{x^2} \frac{1}{t^3} dt$ e) $f(x) = \displaystyle\int_{5}^{e^x} \frac{1}{t^3} dt$

b) $f(x) = \displaystyle\int_{e}^{x} \frac{1}{t} dt$ d) $f(x) = \displaystyle\int_{1}^{x^2} \ln t dt$ f) $f(x) = \left(\displaystyle\int_{1}^{x} \frac{1}{t^2+1} dt\right)^4$

8. Find the area between the $x-$axis and the graph of $f(x) = \dfrac{1}{x}$ between

a) 1 and 2 b) 1 and e

9. Consider the definite integral $\displaystyle\int_{0}^{1} e^{-x^2} dx$. Compute the left and right Riemann sums approximating this integral using a uniform partition of n subintervals where a) $n = 5$ b) $n = 10$

Present your answers as approximations, accurate up to three or more decimal places.

10. Consider the definite integral $\displaystyle\int_{1}^{3} x^2 dx$. Compute the left and right Riemann sums approximating this integral using a uniform partition of n subintervals where $n = 100$. Do not round.

11. Suppose that f is a function with derivative $f'(x) = -2(x+4)^3 (x+2)^2 x (x-3)^4$.

a) Classify the zeroes of f' as maximums, minimums, or points of inflection.

b) Sketch f and f' in the same coordinate system.

c) How many points of inflection does f have?

12. Suppose that $f(x) = (x+5)^3 (x-2)^4$. Find the $x-$coordinates of all points of inflection.

13. Ann and Bethany computed the same indefinite integral. Their final answers look different. Ann's final answer is $\ln|3x - 12| + C$ and Bethay's final answer is $\ln|x-4| + C$. Are the two answers different?

14. A kite 100 ft ft above the ground moves horizontally at a speed of $14 \dfrac{\text{ft}}{\text{s}}$.

a) At what rate is the angle (in radians) between the string and the horizontal decreasing when 200 ft of string have been let out?

b) At what rate is the distance between the kite and its owner increasing when 200 ft of string have been let out?

15. A 5.2 ft tall person is walking away from a 14.8 ft tall street light at a constant rate of $3\,\frac{\mathrm{ft}}{\mathrm{s}}$.

    a) How fast is the tip of his shadow moving when he is 12 feet away from the light?

    b) At what rate is the length of the shadow changing when he is 12 feet away from the light?

16. Suppose that the dimensions of a cylinder change at costant rates: the radius of the base is decreasing at a rate of $1\,\frac{\mathrm{cm}}{\mathrm{min}}$, and its height is increasing at $8\,\frac{\mathrm{cm}}{\mathrm{min}}$. At the moment when the radius is $5\,\mathrm{cm}$ and the height is $25\,\mathrm{cm}$, at what rate are the volume and surface area of the cylinder changing?

17. Car A starts to move at a speed of 40 miles per hour to north. Two hours later, car B startes moving at the same location at a speed of 40 miles per hour to West. At what rate is the distance between the two cars changing

    a) one hour after car B started b) five hours after car B started

18. Consider the function $f(x) = 2x^3 + 4x - 6$.

    a) Prove that f has at least one real zero. b) Prove that f has only one zero.

19. Suppose that $f(x) = x^4 - 10x^3 + 25x^2 + 20x - 100$.

    a) Find the tangent line drawn to the graph at $x = 0$.

    b) This line is a tangent line to the graph of f at two points. Find the $x-$coordinate of the other point of tangency on the tangent line.

20. Two poles, one 8 meters tall and one 12 meters tall, are 15 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles.

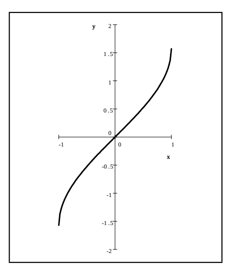
    a) Where should the wire be staked so that the minimum amount of wire is used?

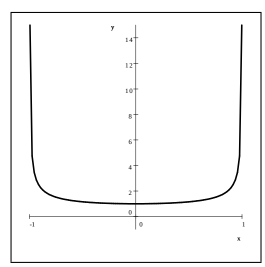
    b) Where should the wire be staked so that the angle formed at the ground by the wire is the greatest?

Answers

1. see handout

2. The domain of $\sin^{-1} x$ is $[-1, 1]$, the domain of $\dfrac{1}{\sqrt{1-x^2}}$ is $(-1, 2)$. $\lim\limits_{x\to 1^-} \dfrac{1}{\sqrt{1-x^2}} = \infty$ indicating that the slope of the tangent lines drawn to the graph of $\sin^{-1} x$ approach ∞, so the tangent line is approaching to be vertical.





3. a) 0 b) $\frac{2}{7}$ c) $\ln 1.5$ d) 2

4. a) 1 b) $h'(x) = 0$ c) For all x, $f^2(x) + g^2(x) = 1$

5. a) $\frac{1}{3}e^{3x+1} + C$ b) $4x + \frac{1}{2}\ln|2x-1| + C$ c) $\tan^{-1} x + C$ d) $\frac{1}{5}\tan^{-1}(5x) + C$ e) $x - \tan^{-1} x + C$

   f) $-\frac{1}{2}\cos 2x + C$ g) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ h) $\frac{1}{7}\sin^{-1}(7x) + C$ i) $-\dfrac{1}{12(2x-5)^6} + C$ j) $\frac{2}{15}(5x+1)^{3/2} + C$

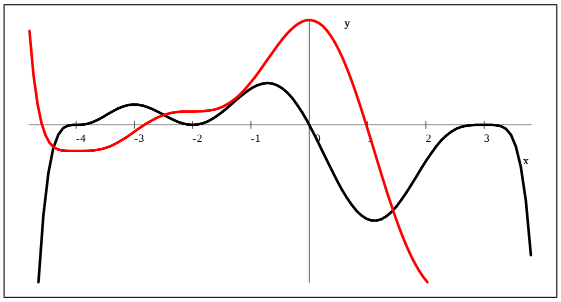
6. a) 0 b) $\ln 2$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{12}$ e) $e - \frac{1}{e}$ f) $\frac{3}{2}$ g) $\frac{\pi}{4}$ h) $\frac{7}{\ln 2}$ i) $\frac{42}{5}$

7. a) $f'(x) = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x$ b) $\frac{1}{x}$ c) $\frac{2}{x^5}$ d) $4x\ln x$ e) e^{-2x} f) $\dfrac{4\left(\tan^{-1} x - \frac{\pi}{2}\right)}{x^2+1}$

8. a) $\ln 2$ b) 1

9. a) $L \approx 0.807\,58$ $R \approx 0.681\,156\,284$ b) $L \approx 0.777\,817$ $R \approx 0.714\,604\,77$ 10. $L = 8.586\,8$ $R = 8.746\,8$

11. a) minimum at $x = -4$, point of inflection at $x = -2$, maximum at $x = 0$, point of inflection at $x = 3$

b) 

c) 5

12. $f'(x) = 7(x+2)(x+5)^2(x-2)^3$

$f''(x) = 42(x+5)(x^2+4x+2)(x-2)^2$

$= 42(x+5)(x+2+\sqrt{2})(x+2-\sqrt{2})(x-2)^2$

$x = -5, -2-\sqrt{2}, -2+\sqrt{2}$

13. The two answers are the same:

$$\ln|3x-12| + C = \ln|3(x-4)| + C = \ln 3 + \ln|x-4| + C = \ln|x-4| + (\ln 3 + C) = \ln|x-4| + C_2$$

14. a) $-\dfrac{7}{200}\dfrac{\text{rad}}{\text{s}} = -0.035\dfrac{\text{rad}}{\text{s}}$ b) $7\sqrt{3}\dfrac{\text{ft}}{\text{s}} \approx 12.124\,356\dfrac{\text{ft}}{\text{s}}$ 15. a) $4.625\dfrac{\text{ft}}{\text{s}}$ b) $1.625\dfrac{\text{ft}}{\text{s}}$

16. $V' = -50\pi\dfrac{\text{cm}^3}{\text{min}}$ $A' = 10\pi\dfrac{\text{cm}^2}{\text{min}}$ 17. a) $16\sqrt{10} \approx 50.596\,44$ b) $\dfrac{240\sqrt{74}}{37} \approx 55.798\,867$

18. a) End-behavior is $-\infty$ and ∞. Then apply the intermediate value theorem.

b) Suppose that there are at least two zeroes. By Rolle's theorem, then there exists between the two roots c with $f'(c) = 0$. But $f'(x) = 6x^2 + 4$ is always positive.

19. a) $y = 20x - 100$ b) 5

20. a) 6 meters from the 8−meter tall pole b) $\sqrt{1446} - 30 \approx 8.026\,3$ meters from the 8−meter tall pole

Last revised: April 24, 2019