Quiz 9 will cover the following material:

All material covered in Classes 1-12

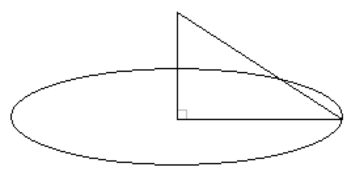
The following Sample Quiz is intended to demonstrate the difficulty level of the questions. It is not intended as a comprehensive review or list of the type of questions that can appear on the quiz.

Sample Quiz 9

1. a) Differentiate $f(x) = \sqrt{1-x^2}$ by computing the limit of the difference quotient.

   b) Use your result to prove that the tangent line drawn to the upper half of the unit circle is perpendicular to the radius drawn to the point of tangency.

2. We create a cone by rotating a right triangle as shown on the picture. If the hypotenuse of the triangle is 1 unit, what is the maximum volume of such a cone? Describe the triangle that guarantees that maximal volume.



3. Compute each ofthe following limits.

   a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$ b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x}$ c) $\displaystyle\lim_{x\to \pi} \frac{\sin x}{x-\pi}$

4. Let $f(x) = x - 3\log_2 x$. Prove that f has at least two zeroes in the interval $[0, 20]$.

5. a) State the least upper bound property of real numbers.

   b) State and prove the first version of the Intermediate Value Theorem

   c) Prove that if a function f is differentiable at a number a, then it is also continuous there.

   d) Prove that $\dfrac{d}{dx}(\sin x) = \cos x$

   e) Prove the sum rule of derivatives.

6. Differentiate each of the following.

   a) $f(x) = \sin x - \dfrac{1}{x}$

   b) $g(x) = x^3 + \sqrt[3]{x} + \dfrac{1}{x^3}$

   c) $f(x) = \dfrac{5x^3 - 2x}{5x^2}$

   d) $y = x^4 - e^4 + \cos x - \dfrac{1}{3}\sin x$

7. Find all values of a and b for which the line $y = -4x + 19$ is a tangent line to $y = -x^3 + ax^2 + bx - 8$ at $x = 3$.

8. Suppose that an object's location function is given by $L(t) = t^3 - 6t^2 + 3t - 10$. Find the moment when the object is moving downward with the greatest speed. What is that greatest speed?

9. Suppose that P is a polynomial with degree 3. Then we can write the polynomial as $P(x) = ax^3 + bx^2 + cx + d$ where a, b, c, and d are real numbers. Find the values of a, b, c, and d if we kow that $P(0) = -2$, $P'(0) = 2$, $P''(0) = 10$ and $P'''(0) = -24$.

Answers

1. see solutions

2. $V = \frac{2\sqrt{3}}{27}\pi$

   The horizontal side is $\frac{\sqrt{3}}{3}$ units long, the vertical side is $\frac{\sqrt{6}}{3}$ units long. One angle is $\tan^{-1}\left(\sqrt{2}\right) \approx 54.736°$

3. a) $\frac{1}{2}$ b) undefined c) -1

4. $f(1) = 1$ and $f(2) = -1$ Therefore, f has a zero in $(1, 2)$ by the intermediate value theorem
   $f(2) = -1$ and $f(16) = 4$ Therefore, f has a zero in $(2, 16)$ by the intermediate value theorem

5. see handouts

6. a) $f'(x) = \cos x + \frac{1}{x^2}$ b) $g'(x) = 3x^2 + \frac{\sqrt[3]{x}}{3x} - \frac{3}{x^4}$ c) $f'(x) = \frac{2}{5x^2} + 1$ d) $y' = 4x^3 - \sin x - \frac{1}{3}\cos x$

7. $a = 3, b = 5$

8. $f'(2) = -9$

9. $a = -4$, $b = 5$, $c = 2$, $d = -2$

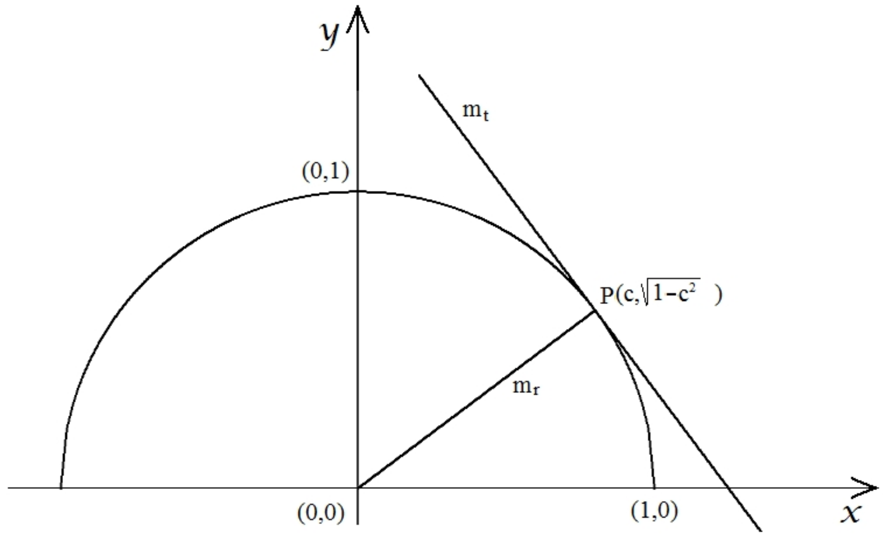
Solution for 1.

1. a) Claim: $\frac{d}{dx}\left(\sqrt{1-x^2}\right) = -\frac{x}{\sqrt{1-x^2}}$

   Proof:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} = \lim_{h\to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} \cdot \frac{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} \\
&= \lim_{h\to 0} \frac{\left[1-(x+h)^2\right] - (1-x^2)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)} = \lim_{h\to 0} \frac{\left[1-\left(x^2+2xh+h^2\right)\right] - \left(1-x^2\right)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0} \frac{1-x^2-2xh-h^2-1+x^2}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)} = \lim_{h\to 0} \frac{-2xh-h^2}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0} \frac{h(-2x-h)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)} = \lim_{h\to 0} \frac{-2x-h}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} \\
&= \frac{-2x}{\sqrt{1-x^2} + \sqrt{1-x^2}} = \frac{-2x}{2\sqrt{1-x^2}} = \boxed{\frac{-x}{\sqrt{1-x^2}}}
\end{aligned}
$$

   b) Let $f(x) = \sqrt{1-x^2}$ - the upper half of the unit circle. Let c be a number between 0 and 1. We will look at the tangent line drawn at the point $\left(c, \sqrt{1-c^2}\right)$.



First, let us figure out the slope of the radius. That can be easily done via the slope formula between $\left(c, \sqrt{1-c^2}\right)$ and $(0,0)$.

$$m_r = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{1-c^2} - 0}{c - 0} = \frac{\sqrt{1-c^2}}{c}$$

Recall that the derivative measures the slope of the tangent line. The derivative of $f(x) = \sqrt{1-x^2}$ is $-\frac{x}{\sqrt{1-x^2}}$. So, the tangent line drawn at $x = c$ will have slope $-\frac{c}{\sqrt{1-c^2}}$ and so

$$m_t = -\frac{c}{\sqrt{1-c^2}}$$

Now the product of the two slopes is

$$m_r \cdot m_t = \frac{\sqrt{1-c^2}}{c} \cdot \left(-\frac{c}{\sqrt{1-c^2}}\right) = -1$$

and so the two lines are perpendicular.