The order of operations rule is an agreement among mathematicians, it simplifies notation. P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong.

P
E
M D
A S

This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.
2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

Example 1. Perform the indicated operations. $20 - 3 \cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20 - 3 \cdot 4 &= && \text{multiplication} \\ 20 - 12 &= && \text{subtraction} \\ &= \boxed{8} \end{aligned}$$

Example 2. Perform the indicated operations. $36 \div 3 \cdot 2$

Solution: it is a very common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first

$$\begin{aligned} 36 \div 3 \cdot 2 &= && \text{division} \\ 12 \cdot 2 &= && \text{multiplication} \\ &= \boxed{24} \end{aligned}$$

Example 3. Perform the indicated operations. $36 \div 2 \div 2$

Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 36 \div 2 \div 2 &= && \text{first division from left} \\ 18 \div 2 &= && \text{division} \\ &= \boxed{9} \end{aligned}$$

Example 4. Perform the indicated operations. $(8-5)^2 - 5 + 2$

Solution: We start with the subtraction within the parentheses. Then we can drop the parentheses.

$$\begin{aligned} (8-5)^2 - 5 + 2 &= 3^2 - 5 + 2 && \text{exponentiation} \\ &= 9 - 5 + 2 && \text{subtraction} \\ &= 4 + 2 && \text{addition} \\ &= \boxed{6} \end{aligned}$$

Notations for Multiplication

There are different ways we denote multiplication. Suppose we wanted to express the multiplication 2 times 3, our options are $\cdot$ (dot), or $\times$ (cross), or *nothing*.

$2 \cdot 3$	or	2×3	or	$(2)(3)$ or $2(3)$ or $(2)3$
This is the preferred notation		We will almost never use this. It's a dinosaur, it is going away....		We will almost never use this with positive numbers

It is a common misconception to believe that parentheses mean multiplication. This is not true. **Parentheses never denote multiplication.** In the expression $2(3)$, the parentheses tell us that we are not looking at the two-digit number 23, but rather at the separate numbers 2 and 3, with *nothing* between them. In written notation, multiplication is the default operation. In other words, if we see two numbers with nothing between them, the *nothing* indicates multiplication. This becomes more clear if we consider expressions such as $2x$ or ab. There is no operation sign (or parentheses) in these expressions and yet we know that the operation is mutiplication. It is the *nothing* between 2 and x and between a and b means multiplication.

Understanding Parentheses

Parentheses have no meaning on their own. They merely help us to understand the context in the written language, like punctuation does in written English. Some parentheses are **grouping symbols** overwriting order of operations. Others clarify boundaries of numbers. For example, it is possible to write $2(3)$ instead of $2 \cdot 3$. The parentheses in $2(3)$ is not a grouping symbol. Rather, it helps us understand that we are not looking at the two-digit number 23. We will refer to this as **clarifying parentheses**. One easy way to tell the difference between the two types of parentheses is that if there is no operation inside a pair of parentheses, then it cannot be a grouping symbol.

We will often see expressions with several pairs of parentheses. Luckily, there are only two possibilities.

$4(3-1)-(7\cdot 2-9)$	$20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$
one pair of parentheses follows the other	one pair of parentheses is nested inside the other

When faced with an order of operations problem, *we must pair the parentheses before starting the computation*. Students are encouraged to use colored pencils to make the pairing more visual such as in $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$. Sometimes authors aim to make notation easier to read by using pairs of parentheses with different shapes, such as [ ] or { }. These different looking parentheses serve as regular grouping symbols, they only made to look differently. For example,
$20-\left(12-2\left[5\cdot 8-6^2\right]+1\right)$ might be easier to read than $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$.

When we have several pairs of parentheses following each other, we perform them left to right. When we are dealing with parentheses nested inside others, we start with the innermost one.

Sometimes we have a case of an **invisible parentheses**. When an operation sign encloses entire expressions, it serves as a grouping symbol. For example, there is no parentheses in sight in the expression $\dfrac{32-4}{9-2}$, and division comes before substraction. However, the division bar stretching over the entire expression indicates that we must perform the subtraction on the top and on the bottom, and only then divide. This becomes obvious if we switch the notation from the division bar to the symbol $\div$.

$$\frac{32-4}{9-2}=(32-4)\div(9-2)$$

In this exmple, the division bar is easier to read, so it will be the notation of choice. Note however, that students often make a mistake when entering things like this in the calculator. The correct answer is 7 but the calculator will give us the wrong answer if we enter it incorrectly as $32-4\div 9-2$. In these cases, the parentheses must become visible.

Example 5. Consider the expression $4\,(3-1)-(7\cdot 2-9)$

a) How many operations are there in the expression?

b) Simplify the expression by applying the order of operations agreement. For each step, write a separate line.

Solution: a) We scan the expression, left to right, and count the operations.

1. multiplication between 4 and $(3-1)$
2. subtraction $3-1$
3. subtraction between the two expressions within the parentheses
4. multiplication between 7 and 2
5. subtracting 9

So there are five operations.

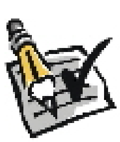
b) We have two pairs of parentheses. We will work them out, left to right. We start with the subtraction $3-1$. Once this subtraction is performed, we no longer need the parentheses. Instead, we will denote the multiplication using the dot notation.

$$
\begin{aligned}
4\,(3-1)-(7\cdot 2-9) &= 4\cdot 2-(7\cdot 2-9) && \text{Now we work on the other parentheses.}\\
& && \text{Multiplication before subtraction}\\
&= 4\cdot 2-(14-9) && \text{Subtraction within parentheses}\\
& && \text{Drop parentheses}\\
&= 4\cdot 2-5 && \text{Multiplication}\\
&= 8-5 && \text{Subtraction}\\
&= \boxed{3}
\end{aligned}
$$

Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

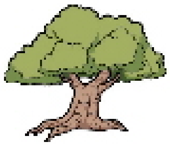
1. $2\cdot 3^2-\left(6^2-2\cdot 5\right)\div 2$
2. $18-7-3$
3. $5^2-2\left(10-2^2\right)$
4. 8^2-3^2
5. $(8-3)^2$
6. $\left(3^3-4\cdot 5+2\right)^2$
7. $\dfrac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1$



Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2\cdot 5^2-\left(6\cdot 5-3^2\right)\div 3$
2. 10^2-7^2
3. $(10-7)^2$
4. $20-7-1$
5. $2^3-\left(11-3^2\right)^2$
6. $\dfrac{5^2-3^2}{2^2}$
7. $\dfrac{(5-3)^2}{2^2}$
8. $120\div 6\cdot 2$
9. $\left((7-4)^2-5\right)^2-1$
10. $\dfrac{22-3^2+2\left(20-3^2-5\right)}{3^2-2^2}$
11. $\dfrac{5+\left(5^2-3^2\right)}{3^2-2\cdot 1^8}$
12. $30-\left(2\left(15-2^3\right)-2^2\right)$
13. $4\left(3\left(2(2^2-1)+1\right)-1\right)+5$
14. $\dfrac{2\left(3^3-4\cdot 5\right)-2^2}{4^2-\left(3^2+2\right)}$



Enrichment

1. Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12 - 2 \cdot 3 - 1^2 + 2 - 3 + 4 = 20$$

Answers

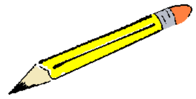
Sample Problems

1. 5   2. 8   3. 13   4. 55   5. 25   6. 81   7. 7

Practice Problems

1. 43   2. 51   3. 9   4. 12   5. 4   6. 4   7. 1   8. 40   9. 15   10. 5   11. 3

12. 20   13. 85   14. 2

Sample Problems  Solutions

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$

   Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent.

$$\begin{aligned} 2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2 &= \quad \text{exponent within parentheses} \\ 2 \cdot 3^2 - (36 - 2 \cdot 5) \div 2 &= \quad \text{multiplication within parentheses} \\ 2 \cdot 3^2 - (36 - 10) \div 2 &= \quad \text{subtraction within parentheses} \\ 2 \cdot 3^2 - (26) \div 2 &= \quad \text{we may drop parentheses now} \\ 2 \cdot 3^2 - 26 \div 2 &= \end{aligned}$$

   Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

$$\begin{aligned} 2 \cdot 3^2 - 26 \div 2 &= \quad \text{exponent} \\ 2 \cdot 9 - 26 \div 2 &= \end{aligned}$$

   Now we execute all multiplications, divisions, left to right

$$\begin{aligned} 2 \cdot 9 - 26 \div 2 &= \quad \text{multiplication} \\ 18 - 26 \div 2 &= \quad \text{division} \\ 18 - 13 &= \quad \text{subtraction} \\ &= \boxed{5} \end{aligned}$$

2. $18 - 7 - 3$

   Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right.

$$\begin{aligned} 18 - 7 - 3 &= \quad \text{first subtraction from left} \\ 11 - 3 &= \quad \text{subtraction} \\ &= \boxed{8} \end{aligned}$$

3. $5^2 - 2\left(10 - 2^2\right)$

   Solution: We start with the parentheses

$$\begin{aligned} 5^2 - 2\left(10 - 2^2\right) &= \quad \text{exponent in parentheses} \\ 5^2 - 2(10 - 4) &= \quad \text{subtraction in parentheses} \\ 5^2 - 2(6) &= \quad \text{drop parentheses} \\ 5^2 - 2 \cdot 6 &= \quad \text{exponents} \\ 25 - 2 \cdot 6 &= \quad \text{multiplication} \\ 25 - 12 &= \quad \text{subtraction} \\ &= \boxed{13} \end{aligned}$$

4. $8^2 - 3^2$

Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned} 8^2 - 3^2 &= \quad \text{first exponent from left} \\ 64 - 3^2 &= \quad \text{exponent} \\ 64 - 9 &= \quad \text{subtraction} \\ &= \boxed{55} \end{aligned}$$

5. $(8-3)^2$

Solution: We start with the parentheses

$$\begin{aligned} (8-3)^2 &= \quad \text{subtraction in parentheses} \\ (5)^2 &= \quad \text{drop parentheses} \\ 5^2 &= \quad \text{exponents} \\ &= \boxed{25} \end{aligned}$$

This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a-b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and the subtract. In $(a-b)^2$ we first subtract b from a and then square the difference.

6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$

Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned} \left(3^3 - 4 \cdot 5 + 2\right)^2 &= \quad \text{exponents within parentheses} \\ (27 - 4 \cdot 5 + 2)^2 &= \quad \text{multiplication within parentheses} \\ (27 - 20 + 2)^2 &= \end{aligned}$$

There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right.

$$\begin{aligned} (27 - 20 + 2)^2 &= \quad \text{subtraction within parentheses} \\ (7+2)^2 &= \quad \text{addition within parentheses} \\ (9)^2 &= \quad \text{drop parentheses} \\ 9^2 &= \quad \text{exponents} \\ &= \boxed{81} \end{aligned}$$

7. $\dfrac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instruct us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}=\left[3+2\left(20-3^2-5\right)\right]\div\left[3^2-2^2\right]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear.

$$\begin{aligned}
\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1 &= && \text{exponent in parentheses}\\
\frac{3+2\left(20-9-5\right)}{3^2-2^2}+4^1 &= && \text{first subtraction from left in parentheses}\\
\frac{3+2\left(11-5\right)}{3^2-2^2}+4^1 &= && \text{subtraction in parentheses}\\
\frac{3+2\left(6\right)}{3^2-2^2}+4^1 &= && \text{drop parentheses}\\
\frac{3+2\cdot 6}{3^2-2^2}+4^1 &= && \text{multiplication on top}\\
\frac{3+12}{3^2-2^2}+4^1 &= && \text{addition on top}\\
\frac{15}{3^2-2^2}+4^1 &=
\end{aligned}$$

Now we work out the bottom, applying order of operations

$$\begin{aligned}
\frac{15}{3^2-2^2}+4^1 &= && \text{first exponent from left to right}\\
\frac{15}{9-2^2}+4^1 &= && \text{exponent}\\
\frac{15}{9-4}+4^1 &= && \text{subtraction}\\
\frac{15}{5}+4^1 &= && \text{same as}\\
15\div 5+4^1 &=
\end{aligned}$$

We now have a division, an addition, and an exponent. We start with the exponent.

$$\begin{aligned}
15\div 5+4^1 &= && \text{exponent, } 4^1=4\\
15\div 5+4 &= && \text{division}\\
3+4 &= && \text{addition}\\
&= \boxed{7}
\end{aligned}$$

