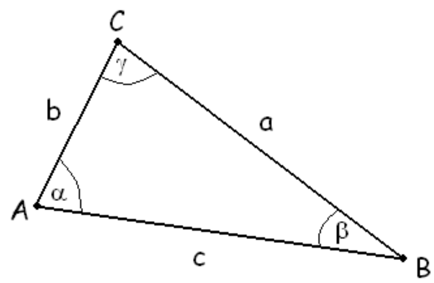
Part 1 – Standard Labeling

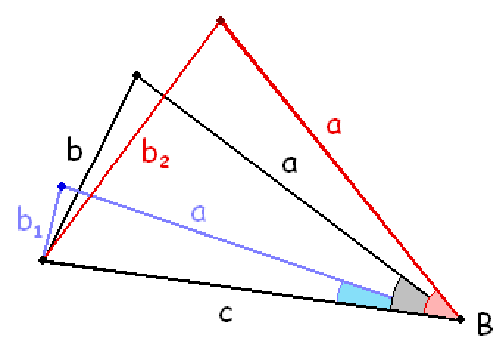
The perimeter of any triangle is simply the sum of the lengths of its three sides. We will have to learn much more for a discussion of the area of right triangles. But before we do that, let us agree first on a method of notation that avoids confusion and lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.

1. **vertices** (singular: vertex)
   Points are usually denoted by uppercase letters. In case of triangles, we often use A, B, and C.
2. **angles**
   Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).
3. **sides**
   Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a, b, and c.



In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A, and side a is always assumed to be the side oppsite to point A and angle α. Point B, angle β, and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondance between side b and angle β.



> **Theorem:** In any triangle ABC, there is a correspondance between the length of a side and the measure of the angle opposite that side:
> The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa. We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

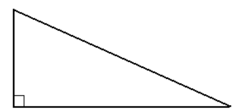
Example 1. Suppose that ABC is a triangle with $\alpha = 82°$ and $\gamma = 39°$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to $180°$. This means that if two angles are given, we can compute the third one. $\beta = 180° - (82° + 39°) = 180° - 121° = 59°$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

There is an easy but important consequence of this property.

Theorem: In any triangle ABC, if two angles have equal measure, then the sides opposite them have equal length. If two sides are equally long, then the angles opposite those sides have equal mesaures. Such a triangle is called **isosceles**.

Definition: A **right triangle** is a triangle with a right angle. A right angle measures $90°$.



We should never assume that a triangle is right, unless it is formally stated either in the problem, or on a picture. Just because an angle appears to be a right angle, it could have a measure of $89.5°$, and that is not right. The measure must be exactly $90°$.

Theorem: In a right triangle, there can only be one right angle and it is the greatest angle in the triangle.

Discussion
1. Find an algebraic and geometric argument for the theorem stated above. Why is it true?
2. What does that mean for the sides of a right triangle?

Definition: In any right triangle, there is a single longest side, and it is opposite of the right angle. This side is called the **hypotenuse** of the right triangle. The shorter sides are called **leg**s.

The hypotenuse of a triangle is often denoted by c, but this is just a convenient habit and not a rule! We should never assume that the rigth angle in a triangle is γ. For all we now, it could be any of α, β, or γ.

We took two pages to establish that there is a longest side in a rigth triangle. The perimeter and are formulas will be very easy.

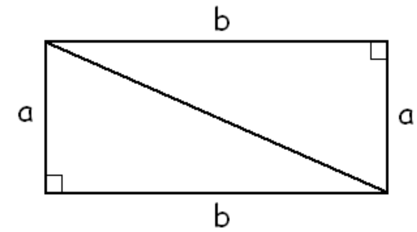
Part 2 - Perimeter and Area Formulas

Recall that the perimeter is just the length of the boundary. This means that the perimeter of any triangle can be computed by adding the lengths of its three sides.

Theorem: The perimeter of a triangle ABC can be computed as $P = a + b + c$.

When we reviewed the area formula for rectangles, we have mentioned that that was a very difficult formula to prove. It is often the case in mathematics that, once we worked very hard for a formula, we use that result over and over. When it comes to area, *all* we know are rectangles. In other words, every area formula was derived from the area formula of the rectangle.

Every right triangle is half of a rectangle. In other words, given any right triangle, we can use two identical copies of it to form a rectangle. We know how to find the area of the rectangle: $A = ab$.
Because the rectangle consists of two identical (also called congruent) right triangles, it naturally follows that each takes up half of the area of the rectangle. Thus the area of the right triangle is $A = \dfrac{ab}{2}$.



Theorem: The area of a right triangle ABC, where c denotes the hypotenuse, is $A = \dfrac{ab}{2}$.

What is unusual about this formula is that we don't need the length of the hypotenuse, only the lengths of the other two sides. This means that given the three sides of a right triangle, we need to know to only use the lengths of the two shorter sides.

Example 2. Compute the perimeter and area of a right triangle with sides $10\,\mathrm{ft}$, $24\,\mathrm{ft}$, and $26\,\mathrm{ft}$.

Solution: The perimeter is just the sum of all three sides.

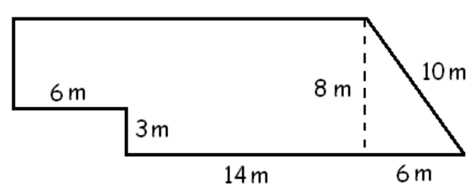
$$P = a + b + c = 10\,\mathrm{ft} + 24\,\mathrm{ft} + 26\,\mathrm{ft} = \boxed{60\,\mathrm{ft}}$$

For the area, we only need the two shorter sides.

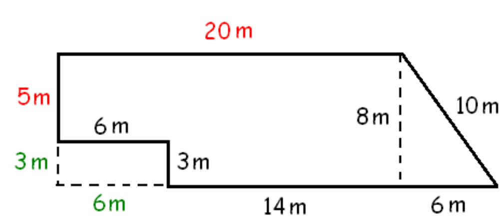
$$A = \frac{ab}{2} = \frac{10\,\mathrm{ft} \cdot 24\,\mathrm{ft}}{2} = \frac{240\,\mathrm{ft}^2}{2} = \boxed{120\,\mathrm{ft}^2}.$$

Although we will derive formulas for the area of more complicated shapes, we can often avoid those by cutting our objects into rectangles and right triangles.

Example 3. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters.



Solution: For the perimeter, we need to find two missing sides. The vertical side is $5\,\mathrm{m}$ long since the opposite sides of rectangles are equally long. Similarly, the missing horizontal side is $20\,\mathrm{m}$ long. We can now compute the perimeter. Notice that the vertical line in the inside is not part of the boundary. We will add all the sides, starting with the long horizontal side.



$$P = 20\,\mathrm{m} + 10\,\mathrm{m} + 6\,\mathrm{m} + 14\,\mathrm{m} + 3\,\mathrm{m} + 6\,\mathrm{m} + 5\,\mathrm{m} = \boxed{64\,\mathrm{m}}$$

Now for the area: We will compute the area of the big rectangle and from it we will subtract the area of the smaller rectangle.

$$A_1 = 8\,\mathrm{m}\cdot 20\,\mathrm{m} - 3\,\mathrm{m}\cdot 6\,\mathrm{m} = 160\,\mathrm{m}^2 - 18\,\mathrm{m}^2 = 142\,\mathrm{m}^2$$

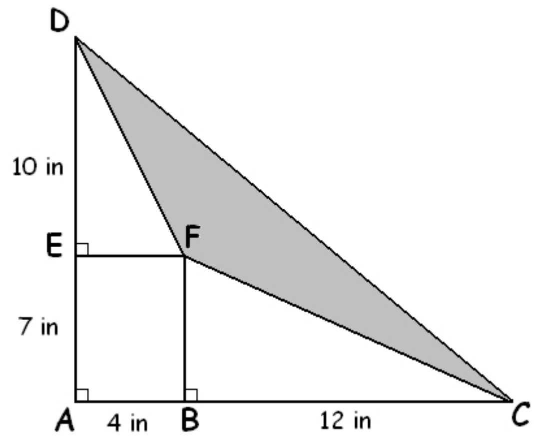
To this we add the area of the right triangle.

$$A_2 = \frac{6\,\mathrm{m}\cdot 8\,\mathrm{m}}{2} = \frac{48\,\mathrm{m}^2}{2} = 24\,\mathrm{m}^2$$

The area of the entire figure is the sum of the two areas:

$$A = A_1 + A_2 = 142\,\mathrm{m}^2 + 24\,\mathrm{m}^2 = \boxed{166\,\mathrm{m}^2}$$

Example 4. Compute area of the shaded region shown on the picture. Units are in inches.



Solution: First we will compute the area of right triangle ACD. $AC = 16\,\mathrm{in}$ and $AD = 17\,\mathrm{in}$.

$$A_{\mathrm{ACD}} = \frac{16\,\mathrm{in}\cdot 17\,\mathrm{in}}{2} = \frac{272\,\mathrm{in}^2}{2} = 136\,\mathrm{in}^2$$

To get the area of the shaded region, we will subtract the areas of the rectangle $ABFE$ and right triangles DEF and BCF.

$$\begin{aligned} A_{\mathrm{ABFE}} &= 7\,\mathrm{in}\cdot 4\,\mathrm{in} = 28\,\mathrm{in}^2 \\ A_{\mathrm{DEF}} &= \frac{10\,\mathrm{in}\cdot 4\,\mathrm{in}}{2} = 20\,\mathrm{in}^2 \\ A_{\mathrm{BCF}} &= \frac{7\,\mathrm{in}\cdot 12\,\mathrm{in}}{2} = \frac{84\,\mathrm{in}^2}{2} = 42\,\mathrm{in}^2 \end{aligned}$$

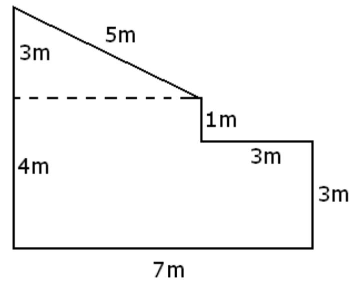
So the shaded area is when we subtract the white areas from the big right triangle.

$$\begin{aligned} A &= A_{\mathrm{ACD}} - (A_{\mathrm{ABFE}} + A_{\mathrm{DEF}} + A_{\mathrm{BCF}}) \\ &= 136\,\mathrm{in}^2 - \left(28\,\mathrm{in}^2 + 20\,\mathrm{in}^2 + 42\,\mathrm{in}^2\right) = 136\,\mathrm{in}^2 - 90\,\mathrm{in}^2 = \boxed{46\,\mathrm{in}^2} \end{aligned}$$

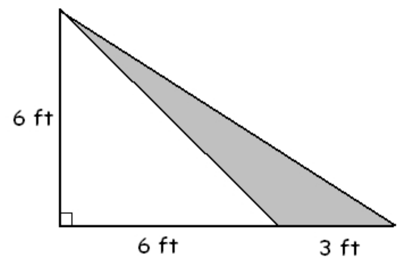
Practice Problems

1. Compute the perimeter and area of a right triangle with sides $7\,\text{ft}$, $25\,\text{ft}$, and $24\,\text{ft}$. Include units in your computation and answer.

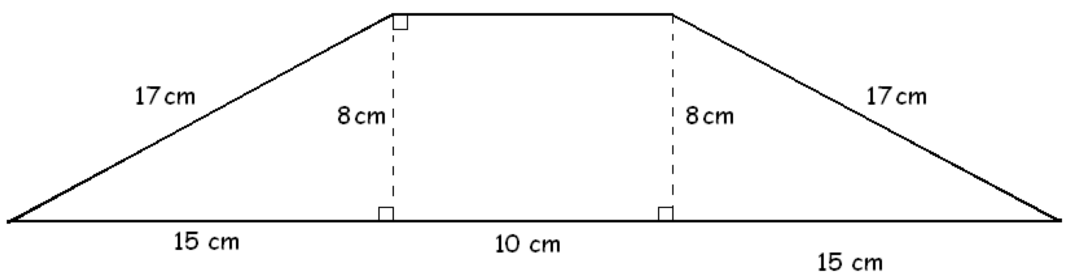
2. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters. Include units in your computation and answer.



3. Compute the area of the shaded region shown on the picture. Include units in your computation and answer.

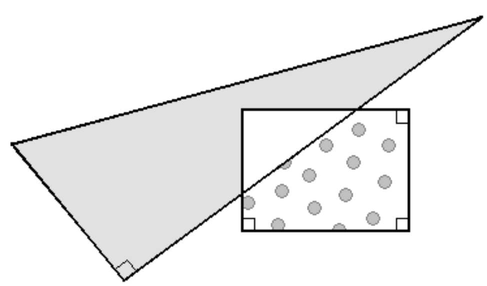


4. Compute the perimeter and area of the figure shown on the picture below. Units are in meters. Include units in your computation and answer.



Enrichment

Consider the figure shown on the picture. The shorter sides of the right triangle are $6\,\text{cm}$ and $10\,\text{cm}$ long. The sides of the rectangle are $5\,\text{cm}$ and $6\,\text{cm}$ long. Which region has the greater area, the shaded or the dotted?



Answers

1. $P = 56\,\text{ft}$, $A = 84\,\text{ft}^2$ 2. $P = 26\,\text{m}$, $A = 31\,\text{m}^2$ 3. $A = 9\,\text{ft}^2$ 4. $P = 84\,\text{cm}$, $A = 200\,\text{cm}^2$

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