Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5

e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem: A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.
A number n is divisible by 3 if the sum of its all digits is divisible by 3.
A number n is divisible by 4 if the two-digit number formed by its last two digits is divisible by 4.
A number n is divisible by 5 if its last digit is 0, or 5.
A number n is divisible by 6 if it is divisible by 2 and by 3.
A number n is divisible by 9 if the sum of its all digits is divisible by 9.
A number n is divisible by 10 if its last digit is 0.
A number n is divisible by 11 if the difference of the sum of digits at odd places and the sum of its digits at even places, is divisible by 11.
A number n is divisible by 12 if it is divisible by 3 and by 4.

Example 2. Consider the numbers 2016, $183\,422$, $606\,060$, $123\,321$, and $38\,115$. Find all numbers from this list that are divisible by

a) 3 b) 9 c) 4 d) 5 e) 11

Solution: a) We need to add all digits. If the sum of the digits is divisible by 3, so is the number.

$2 + 0 + 1 + 6 = 9$. Since 9 is divisible by 3,so is 2016.

$1 + 8 + 3 + 4 + 2 + 2 = 20$. Since 20 is not divisible by 3, neither is $183\,422$.

$6 + 0 + 6 + 0 + 6 + 0 = 18$. Since 18 is divisible by 3, so is $606\,060$.

$1 + 2 + 3 + 3 + 2 + 1 = 12$. Since 12 is divisible by 3, so is $123\,321$.

$3 + 8 + 1 + 1 + 5 = 18$. Since 18 is divisible by 3, so is $38\,115$.

Therefore, the numbers divisible by 3 are: $\boxed{2016,\ 606\,060,\ 123\,321,\ 38\,115}$.

b) The divisibility test for 9 also requires the sum of all digits. We just worked those out.

The sum of digits of 2016 is 9. Since 9 is divisible by 9, so is 2016.

The sum of digits of 183 422 is 20. Since 20 is not divisible by 9, neither is 183 422.

The sum of digits of 606 060 is 18. Since 18 is divisible by 9, so is 606 060.

The sum of digits of 123 321 is 12. Since 12 is not divisible by 9, neither is 123 321.

The sum of digits of 38 115 is 18. Since 18 is divisible by 9, so is 38 115.

Therefore, the numbers divisible by 9 are: $\boxed{2016,\ 606\,060,\ 38\,115}$.

c) The divisibility test for 4 requires to look at the two-digit number formed from the last two digits. If that two-digit number is divisible by 4, so is the number.

2016 ends in 16. Since 16 is divisible by 4, so is 2016.

183 422 ends in 22. Since 22 is not divisible by 4, neither is 183 422.

606 0606 ends in 60. Since 60 is divisible by 4, so is 606 060.

123 321 ends in 21. Since 21 is not divisible by 4, neither is 123 321.

38 115 ends in 15. Since 15 is not divisible by 4, neither is 2016.

Therefore, the numbers divisible by 4 are: $\boxed{2016,\ 606\,060}$.

d) The divisibility test for 5 requires to look at the last digit. If it is 0 or 5, the number is divisible by 5.

Among the numbers 2016, 183 422, 606 060, 123 321, and 38 115, only 606 060 and 38 115 ends in 0 or 5.

Therefore, the numbers divisible by 5 are: $\boxed{606\,060,\ 38\,115}$.

e) The divisibility test for 11 requires to split the digits into two groups, selecting every second one into one group. We add the numbers in each group. If the difference of the two sums is divisible by 11, so is the number.

We split the digits into two groups: $\boxed{2}0\boxed{1}6$. One group is formed from the circled digits, the other group is the rest. The sums in the two groups are $2 + 1 = 3$ and $0 + 6 = 6$. We subtract one from the other. $6 - 3 = 3$. Since 3 is not divisible by 11, neither is 2016.

We split the digits into two groups: $\boxed{1}8\boxed{3}4\boxed{2}2$. The sums in the two groups are $1 + 3 + 2 = 6$ and $8 + 4 + 2 = 14$. We subtract one from the other. $14 - 6 = 8$. Since 8 is not divisible by 11, neither is 183 422.

We split the digits into two groups: $\boxed{6}0\boxed{6}0\boxed{6}0$. The sums in the two groups are $6 + 6 + 6 = 18$ and $0 + 0 + 0 = 0$. We subtract one from the other. $18 - 0 = 18$. Since 18 is not divisible by 11, neither is 606 060.

We split the digits into two groups: $\boxed{1}2\boxed{3}3\boxed{2}1$. The sums in the two groups are $1 + 3 + 2 = 6$ and $2 + 3 + 1 = 6$. We subtract one from the other. $6 - 6 = 0$. Since 0 is divisible by 11, so is 123 321.

We split the digits into two groups: $\boxed{3}8\boxed{1}1\boxed{5}$. The sums in the two groups are $3 + 1 + 5 = 9$ and $8 + 1 = 9$. We subtract one from the other. $9 - 9 = 0$. Since 0 is divisible by 11, so is 38 115.

Therefore, the numbers divisible by 11 are: $\boxed{123\,321,\ 38\,115}$.

Note that in case of divisibility by 11, the difference between the two groups does not need to be zero. Consider, for example 902. The sum of one group is 11, the sum of the other is 0. $11 - 0 = 11$. Since 11 is divisible by 11, so is 902. Indeed, $902 = 11 \cdot 82$

Practice Problems

1. Consider the following numbers: $64,\ 75,\ 80,\ 128,\ 270$

   a) Find all numbers on the list that are divisible by 5.

   b) Find all numbers on the list that are divisible by 3.

   c) Find all numbers on the list that are divisible by 4.

2. Consider the following numbers: $4181,\ 9800,\ 1296,\ 420,\ 55\,050$

   (a) Find all numbers from the list that are divisible by 3.

   (b) Find all numbers from the list that are divisible by 10.

   (c) Find all numbers from the list that are divisible by 6.

3. Consider the following numbers: $28\,072,\ 67\,808,\ 60\,610$, 1296, and 7620.

   (a) Find the numbers from the list that are divisible by 4.

   (b) Find the numbers from the list that are divisible by 5.

   (c) Find the numbers from the list that are divisible by 9.

   (d) Find the numbers from the list that are divisible by 11.

Answers for Practice Problems

1. a) $80, 75, 270$ b) $75, 270$ c) $128, 80, 64$
2. a) $1296,\ 420,\ 55\,050$ b) $9800,\ 420,\ 55\,050$ c) $1296,\ 420,\ 55\,050$
3. a) $28\,072,\ 67\,808,\ 1296,\ 7620$ b) $60\,610,\ 7620$ c) 1296 d) $28\,072, 60\,610$, 7620

