Recall the definition of an improper fraction.

> **Definition:** If the numerator of a fraction is less than its denominator, we call it a **proper fraction**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.
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> If the numerator of a fraction is greater than or equal to its denominator, we call it an **improper fraction**. An improper fraction always expresses a whole unit or more of a quantity. Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

Also recall that if we multiply both numerator and denominator of a fraction by the same non-zero number, the resulting fraction is equivalent to the original fraction. Equivalent fractions express the same amount. We can also write any integer as a fraction. For example, 1 can be written as $\frac{1}{1}$, $\frac{3}{3}$, $\frac{5}{5}$, $\frac{100}{100}$ or 100%. The integer 5 can be written as $\frac{5}{1}$, $\frac{10}{2}$, $\frac{15}{3}$, or 500%. We can also divide both numerator and denominator by the same number. When a fraction's numerator and denominator share no divisor greater than 1, the fraction is **in lowest terms**.

With the introduction of improper fractions, our notation includes a new (and huge) duality. We can look at the expression $\frac{20}{4}$ as a division between two integers, i.e. two objects and an operation. We can also interpret the expression $\frac{20}{4}$ as an improper fraction, i.e. a single object. This ambiguity is only allowed because every question we can ever ask has the same answer, no matter which interpretations we used. As final result, fractions must be presented in their simplest form. For example, the fraction $\frac{15}{18}$ must be reduced to lowest terms and presented as $\frac{5}{6}$. The fraction $\frac{20}{4}$ must be presented as the integer 5 because it is a much simpler presentation than $\frac{5}{1}$. But how do we simplify (if we even can) the improper fraction $\frac{7}{3}$?

The name improper fraction already suggests that there is something wrong with such a fraction. We strongly disagree with this notion. However, this might have not been the general opinion when the concept of mixed numbers was developed.

A dime is a common nickname for the silver colored, small ten-cent coin. 10 dimes are worth a dollar, so one dime can be represented as $\frac{1}{10}$ of a dollar. Suppose we have 42 dimes. How can we express this fact? As an improper fraction, we can of course write $\frac{42}{10}$.

Suppose we go to the bank and change all dimes we can for dollar bills. In this case, we could exchange 40 dimes for four dollar bills. Using this idea, we can write $\frac{42}{10}$ as a mixed number as $4\frac{2}{10}$. The integer part expresses the bills, the fraction part expresses the coins. Of course we can also bring the fraction part to lowest terms and get $4\frac{1}{5}$. For some, this is the only way to present the fraction $\frac{42}{10}$ in its simplest form.

> **Definition:** A **mixed number** is an alternative representation for improper fractions that can not be simplified as integers. A mixed number has an integer part and a fraction part. For example, $2\frac{1}{3}$ is a mixed number where the integer part is 2 and the fraction part is $\frac{1}{3}$.

Not every country uses mixed number notation. In countries that don't use mixed numbers, $2\frac{1}{3}$ would appear as $2+\frac{1}{3}$. In the USA, mixed numbers are fairly common, but their usefullness is debated. Let us also note that this notation is an example where two objects are written next to each other with no operation between them, and it does *not* represent multiplication.

Example 1. Re-write the improper fraction $\frac{87}{10}$ as a mixed number.

Solution: We can think of this as a person with 87 dimes in their pocket. How much money can be exchanged to dollar bills? Since 10 dimes are worth a dollar, we can exchange 80 dimes for eight dollars in paper money, and are left with seven dimes. This can be expressed as $\boxed{8\frac{7}{10}}$.

Example 2. Re-write the improper fraction $\frac{513}{100}$ as a mixed number.

Solution: We can think of this as a person with 403 pennies in their pocket. How much money can be exchanged to dollar bills? Since 100 pennies are worth a dollar, we can exchange 500 pennies for five dollars in paper money, and are left with thirteen pennies. This can be expressed as $\boxed{5\frac{13}{100}}$.

Notice that in converting an improper fraction to a mixed number, we apply division with remainder. The division $87 \div 10 = 8$ R 7 is behind the conversion $\frac{87}{10} = 8\frac{7}{10}$ and the divsion $513 \div 100 = 5$ R 13 is behind the conversion $\frac{513}{100} = 5\frac{13}{100}$. The idea behind mixed numbers is simplifying, which means that the integer part needs to be reduced to lowest terms.

Example 3. Re-write the improper fraction $\frac{28}{10}$ as a mixed number.

Solution: We can think of this as a person with 28 dimes in their pocket. We perform the division with remainder: $28 \div 10 = 2$ R 8. This is the same as saying that $\frac{28}{10} = 2\frac{8}{10}$, just as in the previous examples. However, this mixed number needs to be simplified where we bring the fraction part to lowest terms. Clearly $\frac{8}{10} = \frac{4}{5}$, so $\frac{28}{10} = \boxed{2\frac{4}{5}}$.

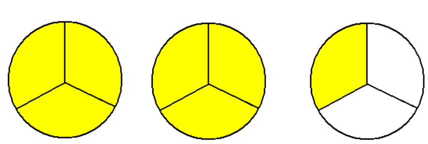
Let us see an example beyond coins. Suppose that we work in a pizza place where each pizza is cut into 6 slices. So, one slice of pizza can be represented as $\frac{1}{6}$. A whole pizza can be represented as 1 (imagine we are not cutting it into slices) or $\frac{6}{6}$ (if we do cut it into slices). How can we express $\frac{25}{6}$ as a mixed number? We perform the division with remainder: $25 \div 6 = 4$ R 1 and we convert $\frac{25}{6}$ to the mixed number $4\frac{1}{6}$. We can interpret this as 4 whole pizzas, and one additional slice. This is correct, 4 whole pizzas can be represented as $4 = \frac{4}{1} = \frac{24}{6}$ that is, 24 slices.

Example 4. Re-write the improper fraction $\frac{17}{6}$ as a mixed number.

Solution: We perform the division with remainder: $17 \div 6 = 2$ R 5. Thus $\frac{17}{6} = \boxed{2\frac{5}{6}}$.

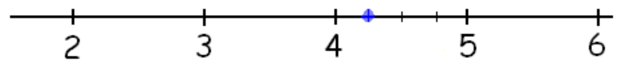
Example 5. Draw a picture representing the improper fraction $\frac{7}{3}$. Use your picture to convert $\frac{7}{3}$ to a mixed number.

Solution: To represent a fraction with denominator three means that we will draw circles and slice them into three equal part. Because this is an improper fraction, we will need more than one circle. We keep starting new units until we have seven slices. Our picture shows that the mixed number corresponding to $\frac{7}{3}$ is $\boxed{2\frac{1}{3}}$.



Example 6. Re-write the improper fraction $\frac{17}{4}$ as a mixed number. Use your result to plot $\frac{17}{4}$ on the number line.

Solution: We perform the divsion with remainder: $17 \div 4 = 4 \text{ R } 1$. Therefore, $\frac{17}{4} = \boxed{4\frac{1}{4}}$. When we plot $4\frac{1}{4}$, we first find 4 and 5. We split the line segment between 4 and 5 into four equal part, and count one from 4.



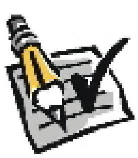
Example 7. Re-write the mixed number $3\frac{2}{5}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into five slices. Then the question is: if someone orders three full pizzas and two more slices, how many slices were ordered? Three full pizzas will account for 15 slices, so all together we have 17 slices. Algebraically, $3 = \frac{3}{1} = \frac{15}{3}$ and so $3\frac{2}{5} = \boxed{\frac{17}{3}}$.

Example 8. Re-write the mixed number $2\frac{5}{8}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into eight slices. Then the question is: if someone orders two full pizzas and five more slices, how many slices were ordered? Two full pizzas will account for 16 slices, so all together we have $16 + 5 = 21$ slices. Algebraically, $2 = \frac{2}{1} = \frac{16}{8}$ and so $2\frac{5}{8} = \boxed{\frac{21}{8}}$.

In this course, we will always have to present fractions in their simplest possible form as final answers. **However, we will not view mixed numbers as more simplified than improper fractions.** This means that improper fractions can always be presented in their improper form as a final answer, as long as it is in lowest terms. For example, $\frac{42}{10}$ is not acceptable as final answer, but $\frac{21}{5}$ is. The mixed number $4\frac{2}{10}$ is not simplified, $4\frac{1}{5}$ is. However, $4\frac{1}{5}$ is not considered more simplified than $\frac{21}{5}$. They are equally acceptable, and, from an algebraic point of view, $\frac{21}{5}$ is actually preferred. We will see that with just a very few exceptions, improper fractions have nicer properties than mixed numbers.



Practice Problems

1. Re-write each of the following improper fractions as mixed numbers. Bring the fraction part to lowest terms.

   a) $\frac{10}{3}$ b) 150% c) $\frac{35}{10}$ d) $\frac{120}{7}$ e) $\frac{42}{8}$ f) $\frac{715}{100}$ g) 320%

2. Re-write each of the mixed numbers as improper fractions in lowest terms.

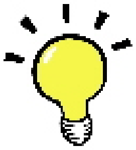
   a) $3\frac{4}{5}$ b) $1\frac{3}{8}$ c) $5\frac{6}{7}$ d) $3\frac{4}{10}$ e) $10\frac{5}{7}$ f) $5\frac{1}{4}$

3. Draw a picture to represent each of the given fractions.

   a) $\frac{13}{4}$ b) $2\frac{3}{5}$ c) $\frac{7}{2}$ d) $\frac{10}{7}$

4. Plot each of the given fractions on the number line.

   a) $\frac{8}{3}$ b) $3\frac{2}{5}$ c) $\frac{9}{2}$ d) $\frac{7}{4}$

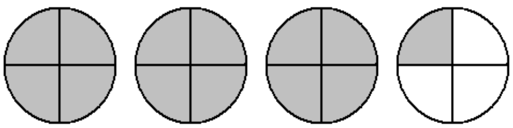


Answers

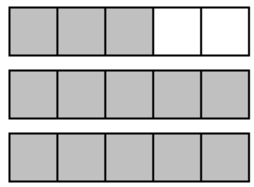
1. a) $3\frac{1}{3}$ b) $1\frac{1}{2}$ c) $3\frac{1}{2}$ d) $17\frac{1}{7}$ e) $5\frac{1}{4}$ f) $7\frac{3}{20}$ g) $3\frac{1}{5}$

2. a) $\frac{19}{5}$ b) $\frac{11}{8}$ c) $\frac{41}{7}$ d) $\frac{17}{5}$ e) $\frac{75}{7}$ f) $\frac{21}{4}$

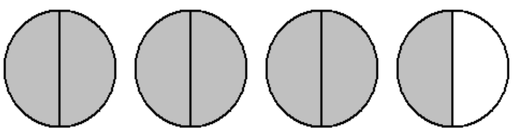
3. a) $\frac{13}{4} = 3\frac{1}{4}$



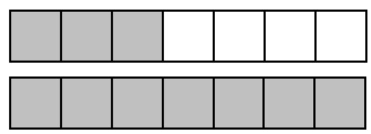
b) $2\frac{3}{5} = \frac{13}{5}$



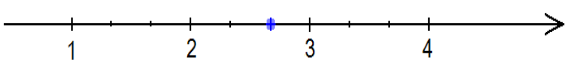
c) $\frac{7}{2} = 3\frac{1}{2}$



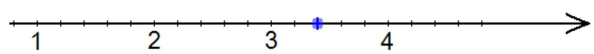
d) $\frac{10}{7} = 1\frac{3}{7}$



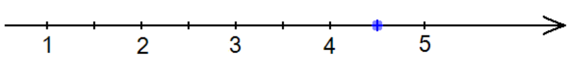
4. a) $\frac{8}{3} = 2\frac{2}{3}$



b) $3\frac{2}{5}$



c) $\frac{9}{2} = 4\frac{1}{2}$



d) $\frac{7}{4} = 1\frac{3}{4}$