Suppose that two natural numbers are given. The greatest common factor and least common multiple of two numbers are important quantities that can be obtained using the prime factorization them. In what follows, we will see how these can be computed.

Definition: Suppose that n and m are any two positive integers. The **greatest common factor** (GCF) of the two numbers is the greatest integer that is a factor of both numbers.

Common factors always exist, because 1 is a common factor of all integers. The greatest common factor (or greatest common divisor) of n and m is denoted by $\gcd(n, m)$.

Example 1. Find the greatest common factor or 60 and 96.

Solution: We can list the factors of 60 and 96.

$$\begin{array}{lcl} \text{factors of } 60 & : & 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 60 \\ \text{factors of } 96 & : & 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96 \end{array}$$

We can list the common factors:

$$\text{common factors:} \quad 1, 2, 3, 4, 6, 12$$

Thus the greatest common factor of 60 and 96 is $\boxed{12}$.

While the method presented above works, it will be increasingly difficult to apply it for larger numbers. There is a more efficient method that uses the prime factorization.

Solution 2: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If d is a divisor of 60, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 96, then d can not have a prime divisor other than 2 and 3. So, common factors can only have 2 and 3 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any factor of 60 can not have more than two $2-$factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32 but not by 64. So, any factor of 96 can not have more than five $2-$factors. In order to be a factor of both, the exponent can be at most 2. We look at the exponents of 3 and in both prime factorizations, there is exactly one $3-$factor. Thus the greatest common factor is

$$2^2 \cdot 3 = \boxed{12}$$

Example 2. Find the greatest common factor or 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If d is a divisor of 240, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 135, then d can not have a prime divisor other than 3 and 5. So, common factors can only have 3 and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 3. 240 has one $3-$factor in its prime factorization. This means that 240 is divisible by 3 but not by 9. So, any factor of 240 can have at most one $3-$factor. 135 has 3^3 in its prime factorization. This means that 135 is divisible by 27 but not by 81. So, any factor of 135 can have at most three $3-$factors. In order to be a factor of both, the exponent can be at most 1. We look at the exponents of 5 and in both prime factorizations, there is exactly one $5-$factor. Thus the greatest common factor is

$$3 \cdot 5 = \boxed{15}$$

Example 3. Find the greatest common factor or 40 and 63.

Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If d is a divisor of 40, then d can not have a prime divisor other than 2 or 5. If d is a divisor of 63, then d can not have a prime divisor other than 3 and 7. So, the common factors can not have any of these in their prime factorization. This means that the two numbers share no divisors besides 1. The greatest common factor of these numbers is $\boxed{1}$. When the greatest common factor of two numbers is 1, we say that the numbers are **relatively prime.**

To compute the greatest common factor of two or more numbers, we list all prime factors that are common to the prime factorization of all numbers. With each such prime number, we use the lowest occuring exponent from the prime factorizations.

Definition: Suppose that n and m are any two positive integers. The **least common multiple** (LCM) of the two numbers is the least positive integer that is a multiple of both numbers.

Common multiples always exist, because the product mn is a common multiple of both integers. The least common multiple of n and m is denoted by $\operatorname{lcm}(n, m)$.

Example 4. Find the greatest common factor or 60 and 96.

Solution: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If T is a multiple of 60, then T must have the prime divisors 2, 3, and 5 in its prime factorization. If T is a multiple of 96, then T must have the prime divisors 2 and 3 in its prime factorization. So, common multiples all contain 2, 3, and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any multiple of 60 must have at least two $2-$factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32. So, any multiple of 96 must have at least five $2-$factors. In order to be a multiple of both, the exponent can be at least 5. We look at the exponents of 3 and in both prime factorizations, there is exactly one $3-$factor. Because 60 is divisible by 5, we need a factor of 5 as well. Thus the least common multiple is

$$2^5 \cdot 3 \cdot 5 = \boxed{480}$$

Example 5. Find the least common multiple or 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If T is a multiple of 240, then T must have 2, 3, and 5 in its prime factorization. If T is a multiple of 135, then T must have 3 and 5 in its prime factorization. So, common multiples must have 2, 3, and 5 in their prime factorization. We now just need to find the exponents of the least common multiple. Furthermore, in order to be a multiple of 240, T must have at least four $2-$factors in its prime factorization. Similarly, in order to be a multiple of 135, T must have at least three $3-$factors in its prime factorization. Thus the least common multiple is

$$2^4 \cdot 3^3 \cdot 5 = \boxed{2160}$$

Example 6. Find the greatest common factor or 40 and 63.

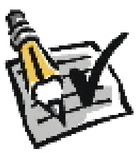
Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If T is a multiple of 40, then T must have 2 and 5 in its prime factorization. If T is a multiple of 63, then T must have 3 and 7 in its prime factorization. So, the common multiples must have 2, 3, 5, and 7 in its prime factorization. For the exponents: T must have at least three 2−factors if it is a multiple of 40 and T must have at least two 3−factors if it is a multiple of 63. Thus the least common multiple is

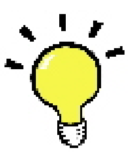
$$2^3 \cdot 3^2 \cdot 5 \cdot 7 = \boxed{2520}$$

Discussion: Consider our results for examples 1-6. In each case, find the product of the two numbers and the product of their least common multiple and greatest common factor. For example, in case of 60 and 96, $\operatorname{lcm}(60, 96) = 480$ and $\gcd(60, 96) = 12$. So, $60 \cdot 96 = 5760$ and $480 \cdot 12 = 5760$. Is this a coincidence that the products are the same?



Practice Problems

1. Find the greatest common factor and least common multiple of each of the given pairs of numbers.
   a) 72 and 300 b) 240 and 3600 c) 100 and 420
2. Is it possible that the greatest common factor and least common multiple of two numbers are equal to each other?
3. Find all possible positive integer values of n if we know that $\gcd(6, 120, n) = 6$ and $\operatorname{lcm}(6, 120, n) = 120$.



Answers

1. a) 12 and 1800 b) 240 and 3600 c) 20 and 2100
2. Yes, it is possible. For example, $\gcd(4, 4) = 4$ and $\operatorname{lcm}(4, 4) = 4$. But if $m \neq n$, then the two will never be equal.
3. $6, 12, 24, 30, 60, 120$

Solution: $\gcd(6, 120, n) = 6$ implies that 6 is a divisor of n.
Similarly, $\operatorname{lcm}(6, 120, n) = 120$ implies that n is a divisor of 120
all divisors of 120 are: $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$
all divisors of 120 that are divisible by 6 are: $\boxed{6, 12, 24, 30, 60, 120}$ - and all these numbers work.

