Factoring by the AC-method is extremely useful because it adresses the most difficult situation, factoring a general quadratic expression with a leading coefficient, such as $6x^2 - 5x - 4$. Trial and error still works, but it is more difficult because there are more possibilities to be considered. The AC-method is a neat and powerful method to quickly factor a trinomial. The main steps are: we cleverly take apart the linear term into two parts and then factor by grouping.

In case of a general quadratic equation, we often denote the coefficients by a, b, and c, where the trinomial is $ax^2 + bx + c$. Notice the addition. This means that coefficients also carry the negative signs if there are any. In the case of $6x^2 - 5x - 4$, $a = 6$, $b = -5$, and $c = -4$.

Example 1. Completely factor the expression $6x^2 - 5x - 4$.

Solution: We will use the AC-method. The first step is to re-write the middle term, $-5x$ into a sum of two terms in such a way that grouping would work after that. To do that, we need to find two numbers p and q with a sum of -5 and a product that is the same as the product ac (hence the name, AC-method.) In this particular case, $ac = 6(-4) = -24$. So we need to find two numbers, p and q, such that

$$p + q = -5 \text{ and } pq = -24$$

This sounds very similar to the trial and error method. Indeed, we will apply the same method to find p and q. A negative product pq indicates that one of p and q is positive and the other is negative. The negative sum -5 indicates that between the two numbers p and q, the negative one has a greater absolute value. These observations will make our task much easier. There are infinitely many integer pair solutions for $p + q$. There are just a few ways to solve the second equation for integers, and so we start there.

We list all the pairs of positive numbers with a product of 24, and take the opposite of the greater one in each pair. We are looking for the pair with sum -5.

-24	
1	-24
2	-12
3	-8
4	-6

Now we consider these pairs as candidates for p and q. We are looking for the pair with sum -5. Clearly that is -8 and 3. Once we found p and q with product -24 and sum -5, we can rewrite the middle term, $-5x$ as $-8x + 3x$ or $3x - 8x$ and factor by grouping.

$$\begin{aligned} 6x^2 - 5x - 4 &= 6x^2 - 8x + 3x - 4 \\ &= 2x(3x - 4) + 1(3x - 4) \\ &= (2x + 1)(3x - 4) \end{aligned}$$

and so the factored form is $\boxed{(2x + 1)(3x - 4)}$. We can check our solution by multiplying back: $(2x + 1)(3x - 4) = 6x^2 - 8x + 3x - 4 = 6x^2 - 5x - 4$ and so our solution is correct.

Example 2. Completely factor the expression $10x^2 - 19x + 6$.

Solution: In this case, $a = 10$, $b = -19$, and $c = 6$. The product ac is 60. We are looking for two numbers p and q with a product of 60 and a sum of -19. A positive product indicates that both p and q are positive, or both of them negative. The negative sum indicates that both p and q are negative.

As always, we start with $pq = 60$. We list all the pairs of negative numbers with a product of 60.

60	
-1	-60
-2	-30
-3	-20
-4	-15
-5	-12
-6	-10

The only pair with a sum -19 is -15 and -4. Now we know how to re-write the linear term, $-19x$ and then factor by grouping.

$$\begin{aligned} 10x^2 - 19x + 6 &= 10x^2 - 15x - 4x + 6 && \text{factor by grouping} \\ &= 5x(2x-3) - 2(2x-3) \\ &= (5x-2)(2x-3) \end{aligned}$$

Therefore, $10x^2 - 19x + 6 = \boxed{(5x-2)(2x-3)}$. We check by multiplying back:

$(5x-2)(2x-3) = 10x^2 - 15x - 4x + 6 = 10x^2 - 19x + 6$ and so our solution is correct.

What happens if we cannot find a suitable pair of numbers p and q? If that is the case, the trinomial cannot be factored using integer coefficients. At this point, we will just say that the expression is **prime** or **irreducible**. We are still responsible however, to factor out the GCF if there is any.

Example 3. Completely factor the expression $3x^2 - 4x + 2$.

Solution: In this case, $ac = 6$. Therefore, we are looking for two numbers p and q with a product of 6 and a sum -4. A positive product indicates that both p and q are negative or both are positive. The negative sum -4 indicates that both p and q are negative.

We start with the equation $pq = 6$. We list all the pairs of negative numbers with a product of 6.

6	
-1	-6
-2	-3

The sum of the first pair is -7 and the sum of the second pair is -5. There are no pairs with sum -4.

This indicates that the trinomial cannot be factored. Therefore, our answer is $\boxed{3x^2 - 4x + 2}$ for the factored form.

Sometimes the product ac is too rich in divisors to list all pairs. If the second term has a small coefficient, there is a neat shortcut to find the righ p and q. Th following example illustrates this trick.

Example 4. Completely factor the expression $24x^2 + x - 10$.

Solution: In this case, $ac = -240$. We are looking for p and q with $pq = -240$ and $p + q = 1$. The number -240 has quite a number of divisors, so we will find the right pair without listing all the pairs. This trick is not possible for all cases, but it will work here. The negative product pq indicates that one of p and q is positive and the other is negative. The positive sum 1 indicates that between the two numbers p and q, the positive one has a greater absolute value.

When we add a positive and a negative number, we subtract the absolute values. Considering the absolute values $|p|$ and $|q|$, their product is 240 and their difference is 1. That means that the right pair of factors are very close to each other. Consequently, both numbers must be fairly close to the square root of 240. We enter $\sqrt{240}$ into our calculator. This is not an integer, the calculator shows $\sqrt{240} = 15.491\,933\ldots$ If two numbers are close to each other and their product is 240, then they both are fairly close to 15. So we round up to 16 and start looking for divisors counting backward: 16, 15, 14, etc. Except, we won't even reach 14 because we immediately bump into 15 and 16. Our p and q is 16 and -15. We can now easily factor by grouping.

$$\begin{aligned} 24x^2 + x - 10 &= 24x^2 + 16x - 15x - 10 \\ &= 8x\,(3x+2) - 5\,(2x+2) = (8x-5)\,(3x+2) \end{aligned}$$

Example 5. One side of a rectangle is 2 feet shorter than three times another side. Find the sides of the rectangle if we also know that its area is $176\,\mathrm{ft}^2$.

Solution: If we label one side as x, then the other side can be written as $3x - 2$. The equation will express the area of the rectangle.

$$\begin{aligned} x\,(3x-2) &= 176 \\ 3x^2 - 2x &= 176 \\ 3x^2 - 2x - 176 &= 0 \end{aligned}$$

We will factor this trinomial by the AC-method. In this case, $ac = 3\,(-176) = -528$. To factor $3x^2 - 2x - 176$, we need to find two integers p and q with product -528 and sum -2. The negative product indicates that one of p and q is positive, the other one is negative. Therefore, the difference between their absolute values is 2. That is fairly small for a large product such as -528. If the two numbers multiplying each other to 528 are close to each other, they also must be close to $\sqrt{528} = 22.978\,251\ldots$ We roll up to 23 and start looking for pairs of integers multiplying to 528. We will try 23, 22, 21, and so on. We almost immediately find 22 and 24. Therefore p and q are -24 and 22. We can now easily factor the trinomial.

$$\begin{aligned} 3x^2 - 2x - 176 &= 3x^2 - 24x + 22x - 176 \\ &= 3x\,(x-8) + 11\,(2x-8) \\ &= (3x+11)\,(x-8) \end{aligned}$$

Applying the zero product rule, we solve $3x + 11 = 0$ and $x - 8 = 0$ ad obtain $x = -\dfrac{11}{3}$ or $x = 8$. Since x represents a distance and distances cannot be negative, we rule out $-\dfrac{11}{3}$ and are left with $x = 8$. If x is 8, then $3x - 2 = 3 \cdot 8 - 2 = 22$. Thus the sides of the rectangle are $\boxed{8\,\mathrm{ft} \text{ and } 22\,\mathrm{ft}}$ long.

We check: 22 is indeed two less than three times 8, and the area of the rectangle is $8\,\mathrm{ft}\,(22\,\mathrm{ft}) = 176\,\mathrm{ft}^2$, and so our solution is correct.

Quadratic equations often have two solutions. In the previous example, one solution of the equation was easily ruled out, but that is not always the case. Often times both solutions of the equation result in a meaningful solution. The next example illustrates this.

Example 6. Twice the square of a number is 35 greater than three times the number. Find all such numbers.

Solution: If we label this number by x, then the square of this number is x^2 and twice the square is $2x^2$. The equation will compare twice the square of the number and three times the number.

$$\begin{aligned} 2x^2 &= 3x+35 \\ 2x^2-3x-35 &= 0 \end{aligned}$$

We will factor $2x^2-3x-35$. The product ac is -70. Therefore, we are lookin for two integers p and q with product -70 and sum -3. These are easily found: -10 and 7. We now know how to take apart the middle term before grouping.

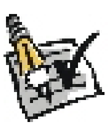
$$\begin{aligned} 2x^2-3x-35 &= 0 \\ 2x^2-10x+7x-35 &= 0 \\ 2x\left(x-5\right)+7\left(x-5\right) &= 0 \\ \left(2x+7\right)\left(x-5\right) &= 0 \end{aligned}$$

We solve the linear equations $2x+7=0$ and $x-5=0$ and obtain $x=-\dfrac{7}{2}$ and $x=5$. We check both:

If the number is 5, then twice its square is $2\cdot 5^2=50$, and three times the number is 15. Indeed, 50 is 35 greater than 15. So 5 works.

If the number is $-\dfrac{7}{2}$, then twice its square is $2\left(-\dfrac{7}{2}\right)^2=2\cdot\dfrac{49}{4}=\dfrac{49}{2}$. Three times the number is $3\left(-\dfrac{7}{2}\right)=-\dfrac{21}{2}$. The difference between $\dfrac{49}{2}$ and $-\dfrac{21}{2}$ is $\dfrac{49}{2}-\left(-\dfrac{21}{2}\right)=\dfrac{49+21}{2}=\dfrac{70}{2}=35$.

We found that both $\boxed{-\dfrac{7}{2}\text{ and }5}$ works.



Practice Problems

1. Completely factor each of the following.

   a) $30x-15y+6ax-3ay$
   b) $xy-y-x+1$
   c) $6a^2b^2-4a^2bc-10a^2c^2$
   d) $a^2m+2a^2n-b^2m-2b^2n$
   e) $x^2-4y^2+m^2x^2-4m^2y^2$
   f) b^2-a+ab^2-1
   g) $2m^2-18n^4+2m^2p^2-18n^4p^2$
   h) $a^2x^2-a^2y^2+b^2x^2-b^2y^2$
   i) $3x^2-2x-1$
   j) $6x^2-5x+1$
   k) $14x-12x^2+10$
   l) $5m^2-7mn+2n^2$
   m) $29px-21p^2+10x^2$
   n) $2x^4-3y^4+x^2y^2$

2. Solve each of the following equations.

   a) $6x^4-x^3=2x^2$ b) $5a^2+5=26a$ c) $11p+35p^2=6$

3. One side of a rectangle is $4\ \text{in}$ shorter than 3 times the other side. Find the sides of the rectangle if its area is $319\ \text{in}^2$.

Answers

Practice Problems

1. a) $3(a+5)(2x-y)$ b) $(x-1)(y-1)$ c) $2a^2(b+c)(3b-5c)$ d) $(a-b)(a+b)(m+2n)$
   e) $(x-2y)(x+2y)\left(m^2+1\right)$ f) $(b-1)(b+1)(a+1)$ g) $-2\left(3n^2-m\right)\left(3n^2+m\right)\left(p^2+1\right)$
   h) $(x-y)(x+y)\left(a^2+b^2\right)$ i) $(3x+1)(x-1)$ j) $(3x-1)(2x-1)$
   k) $-2(2x+1)(3x-5)$ l) $(5m-2n)(m-n)$ m) $(5x-3p)(2x+7p)$
   n) $(x-y)(x+y)\left(2x^2+3y^2\right)$
2. a) $-\frac{1}{2}, 0, \frac{2}{3}$ b) $\frac{1}{5}, 5$ c) $-\frac{3}{5}, \frac{2}{7}$
3. 11 in by 29 in