We have seen the difference of squares theorem that plays a fundamental role in factoring. Let us first recall the theorem.

Theorem: (The Difference of Squares Theorem) For any quantities A and B,

$$A^2 - B^2 = (A+B)(A-B)$$

Notice that $A+B$ and $A-B$ on the right-hand side are conjugates. The identical terms and alternating signs in front of B cause a cencellation between O and I in FOIL.

$$\begin{aligned}(A+B)(A-B) &= A^2 - AB + AB - B^2 \qquad\qquad -AB+AB=0 \\ &= \boxed{A^2 - B^2}\end{aligned}$$

This quite mechanical idea can be generalized to other products. Consider the following products.

Example 1. Expand each of the following.

a) $(A-B)\left(A^2+AB+B^2\right)$ b) $(A-B)\left(A^3+A^2B+AB^2+B^3\right)$

Solution: a) We will apply the distributive law.

$$\begin{aligned}(A-B)\left(A^2+AB+B^2\right) &= A\left(A^2+AB+B^2\right) - B\left(A^2+AB+B^2\right) \\ &= A^3+A^2B+AB^2-\left(A^2B+AB^2+B^3\right) \\ &= A^3+A^2B+AB^2-A^2B-AB^2-B^3 \\ &= \boxed{A^3-B^3}\end{aligned}$$

b) Similarly,

$$\begin{aligned}(A-B)\left(A^3+A^2B+AB^2+B^3\right) &= A\left(A^3+A^2B+AB^2+B^3\right) - B\left(A^3+A^2B+AB^2+B^3\right) \\ &= A^4+A^3B+A^2B^2+AB^3-A^3B-A^2B^2-AB^3-B^4 \\ &= \boxed{A^4-B^4}\end{aligned}$$

This pattern goes on with higher and higher exponents. Indeed, $A^n - B^n$ can be factored for every natural number $n \geq 2$.

Theorem: For any quantities A and B,

$$\begin{aligned}A^3 - B^3 &= (A-B)\left(A^2+AB+B^2\right) \\ A^4 - B^4 &= (A-B)\left(A^3+A^2B+AB^2+B^3\right) \\ A^5 - B^5 &= (A-B)\left(A^4+A^3B+A^2B^2+AB^3+B^4\right) \\ &\vdots \\ A^n - B^n &= (A-B)\left(A^{n-1}+A^{n-2}B+A^{n-3}B^2+\ldots+A^2B^{n-3}+AB^{n-2}+B^{n-1}\right)\end{aligned}$$

We will be focusing on the difference of cubes theorem:

$$A^3 - B^3 = (A-B)\left(A^2+AB+B^2\right)$$

We can not verify this yet, but the second, longer factor, A^2+AB+B^2 is irreducible, it can not be factored.

Example 2. Factor each of the following.

a) $x^3 - 125y^9$ b) $8p^3 - 27q^{12}$

Solution: a) We will factor the expression applying the difference of cubes theorem. A will be 'played' by x and B by $5y^3$, because $x^3 - 125y^9 = x^3 - \left(5y^3\right)^3$.

$$\begin{aligned} A^3 - B^3 &= (A-B)\left(A^2 + AB + B^2\right) \quad \text{where } A = x \quad \text{and} \quad B = 5y^3 \\ x^3 - \left(5y^3\right)^3 &= \left(x - 5y^3\right)\left(x^2 + x\left(5y^3\right) + \left(5y^3\right)^2\right) \\ &= \boxed{\left(x - 5y^3\right)\left(x^2 + 5xy^3 + 25y^6\right)} \end{aligned}$$

b) Similarly, $A = 2p$ and $B = 3q^4$, because $8p^3 - 27q^{12} = (2p)^3 - \left(3q^4\right)^3$

$$\begin{aligned} A^3 - B^3 &= (A-B)\left(A^2 + AB + B^2\right) \quad \text{where } A = 2p \quad \text{and} \quad B = 3q^4 \\ (2p)^3 - \left(3q^4\right)^3 &= \left(2p - 3q^4\right)\left((2p)^2 + (2p)\left(3q^4\right) + \left(3q^4\right)^2\right) \\ &= \boxed{\left(2p - 3q^4\right)\left(4p^2 + 6pq^4 + 9q^8\right)} \end{aligned}$$

Recall that the sum of two squares can never be factored. However, this is not the case with cubes, or fifth powers or seven powers etc. Consider the following products.

Example 3. Expand each of the following.

a) $(A+B)\left(A^2 - AB + B^2\right)$ b) $(A+B)\left(A^4 - A^3B + A^2B^2 - AB^3 + B^4\right)$

Solution: a) We will apply the distributive law.

$$\begin{aligned} (A+B)\left(A^2 - AB + B^2\right) &= A\left(A^2 - AB + B^2\right) + B\left(A^2 - AB + B^2\right) \\ &= A^3 - A^2B + AB^2 + A^2B - AB^2 + B^3 \\ &= \boxed{A^3 + B^3} \end{aligned}$$

b) $(A+B)\left(A^4 - A^3B + A^2B^2 - AB^3 + B^4\right) =$

$$\begin{aligned} &= A\left(A^4 - A^3B + A^2B^2 - AB^3 + B^4\right) + B\left(A^4 - A^3B + A^2B^2 - AB^3 + B^4\right) \\ &= A^5 - A^4B + A^3B^2 - A^2B^3 + AB^4 + A^4B - A^3B^2 + A^2B^3 - AB^4 + B^5 \\ &= \boxed{A^5 + B^5} \end{aligned}$$

This pattern goes on with higher and higher exponents. Indeed, $A^n + B^n$ can be factored for every odd natural number $n \geq 3$.

Theorem: For any quantities A and B, and odd natural number n,

$$\begin{aligned} A^3 + B^3 &= (A+B)\left(A^2 - AB + B^2\right) \\ A^5 + B^5 &= (A+B)\left(A^4 - A^3B + A^2B^2 - AB^3 + B^4\right) \\ A^7 + B^7 &= (A+B)\left(A^6 - A^5B + A^4B^2 - A^3B^3 + A^2B^4 - AB^5 + B^6\right) \\ &\vdots \\ A^n + B^n &= (A+B)\left(A^{n-1} - A^{n-2}B + A^{n-3}B^2 - \ldots + A^2B^{n-3} - AB^{n-2} + B^{n-1}\right) \end{aligned}$$

We will be focusing on the sum of cubes theorem:

$$A^3 + B^3 = (A + B)\left(A^2 - AB + B^2\right)$$

Similar to the difference of cubes, the second, longer factor, $A^2 - AB + B^2$ is irreducible, it can not be factored.

Example 4. Factor each of the following.

a) $x^3 + 125y^9$ b) $(x + 2)^3 + 27$

Solution: a) We will factor the expression applying the sum of cubes theorem. A will be 'played' by x and B by $5y^3$, because $x^3 + 125y^9 = x^3 + \left(5y^3\right)^3$.

$$\begin{aligned} A^3 + B^3 &= (A + B)\left(A^2 - AB + B^2\right) \quad \text{where } A = x \quad \text{and} \quad B = 5y^3 \\ x^3 + \left(5y^3\right)^3 &= \left(x + 5y^3\right)\left(x^2 - x\left(5y^3\right) + \left(5y^3\right)^2\right) \\ &= \boxed{\left(x + 5y^3\right)\left(x^2 - 5xy^3 + 25y^6\right)} \end{aligned}$$

b) Similarly, $A = x + 2$ and $B = 3$, because $27 = 3^3$. A will be 'played' by $x + 2$ and B by 3.

$$\begin{aligned} A^3 + B^3 &= (A + B)\left(A^2 - AB + B^2\right) \quad \text{where } A = x + 2 \quad \text{and} \quad B = 3 \\ (x + 2)^3 + 3^3 &= (x + 2 + 3)\left((x + 2)^2 - (x + 2)(3) + 3^2\right) \\ &= (x + 5)\left(x^2 + 4x + 4 - 3x - 6 + 9\right) \\ &= \boxed{(x + 5)\left(x^2 + x + 7\right)} \end{aligned}$$

Sample Problems

Completely factor each of the following.

1. $x^3 - 8y^3$
2. $125 - 27a^{12}$
3. $1000 + x^6$
4. $(x + 1)^3 - 27$
5. $-2a^7 - 2a^4b^9$
6. $(a + 2)^3 + (a - 2)^3$
7. $a^6 - b^6$

Practice Problems

Completely factor each of the given expressions.

1. $8x^3 + 1000$
2. $(q + 10)^3 + q^3$
3. $m^3 - (y + 1)^3$
4. $a^5b - a^2bx^6$
5. $4a^3m + 2a^3n - 4b^3m - 2b^3n$
6. $3n^2x^3 - 12m^2y^3 - 12m^2x^3 + 3n^2y^3$
7. $2a^5 - 2a^2 - 2a^2b^2 + 2a^5b^2$
8. $(3a - 1)^3 + (a - 3)^3$
9. $(3a - 1)^3 - (a - 3)^3$
10. $2 - 2x^6$

Answers

Sample Problems

1. $(x-2y)\left(x^2+2xy+4y^2\right)$ 2. $-\left(3a^4-5\right)\left(9a^8+15a^4+25\right)$ 3. $\left(x^2+10\right)\left(x^4-10x^2+100\right)$

4. $(x-2)\left(x^2+5x+13\right)$ 5. $-2a^4\left(a+b^3\right)\left(a^2-ab^3+b^6\right)$ 6. $2a\left(a^2+12\right)$

7. $(a+b)(a-b)\left(a^2+ab+b^2\right)\left(a^2-ab+b^2\right)$

Practice Problems

1. $8(x+5)\left(x^2-5x+25\right)$ 2. $2(q+5)\left(q^2+10q+100\right)$ 3. $(m-y-1)\left(m^2+my+m+y^2+2y+1\right)$

4. $a^2b\left(a-x^2\right)\left(a^2+ax^2+x^4\right)$ 5. $2(a-b)\left(a^2+ab+b^2\right)(2m+n)$

6. $-3(x+y)\left(x^2-xy+y^2\right)(2m-n)(2m+n)$ 7. $2a^2(a-1)\left(a^2+a+1\right)\left(b^2+1\right)$

8. $4(a-1)\left(7a^2-2a+7\right)$ 9. $2(a+1)\left(13a^2-22a+13\right)$ 10. $-2(x-1)(x+1)\left(x+x^2+1\right)\left(x^2-x+1\right)$

Sample Problems Solutions

Completely factor each of the following.

1. $x^3 - 8y^3$

   Solution: We will factor via the difference of cubes theorem,
   $A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$. In this case, $A = x$ and $B = 2y$.

$$\begin{aligned} x^3 - 8y^3 &= x^3 - (2y)^3 = (x - (2y))\left(x^2 + x(2y) + (2y)^2\right) \\ &= (x - 2y)\left(x^2 + 2xy + 4y^2\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} (x - 2y)\left(x^2 + 2xy + 4y^2\right) &= x\left(x^2 + 2xy + 4y^2\right) - 2y\left(x^2 + 2xy + 4y^2\right) \\ &= x^3 + 2x^2y + 4xy^2 - 2x^2y - 4xy^2 - 8y^3 = x^3 - 8y^3 \end{aligned}$$

2. $125 - 27a^{12}$

   Solution: We first factor out -1.

$$125 - 27a^{12} = -\left(27a^{12} - 125\right)$$

   We will now factor via the difference of cubes theorem,
   $A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$. In this case, $A = 3a^4$ and $B = 5$.

$$\begin{aligned} 125 - 27a^{12} &= -\left(27a^{12} - 125\right) = -\left(\left(3a^4\right)^3 - 5^3\right) \\ &= -\left(\left(3a^4\right) - 5\right)\left(\left(3a^4\right)^2 + \left(3a^4\right)5 + 5^2\right) = -\left(3a^4 - 5\right)\left(9a^8 + 15a^4 + 25\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} -\left(3a^4 - 5\right)\left(9a^8 + 15a^4 + 25\right) &= -\left(3a^4\left(9a^8 + 15a^4 + 25\right) - 5\left(9a^8 + 15a^4 + 25\right)\right) \\ &= -\left(27a^{12} + 45a^8 + 75a^4 - 45a^8 - 75a^4 - 125\right) \\ &= -\left(27a^{12} - 125\right) = -27a^{12} + 125 = 125 - 27a^{12} \end{aligned}$$

3. $1000 + x^6$

   Solution: We will factor via the sum of cubes theorem,
   $A^3 + B^3 = (A + B)\left(A^2 - AB + B^2\right)$.

$$\begin{aligned} 1000 + x^6 &= x^6 + 1000 = \left(x^2\right)^3 + 10^3 \\ &= \left(x^2 + 10\right)\left(\left(x^2\right)^2 - 10x^2 + 10^2\right) = \left(x^2 + 10\right)\left(x^4 - 10x^2 + 100\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} \left(x^2 + 10\right)\left(x^4 - 10x^2 + 100\right) &= x^2\left(x^4 - 10x^2 + 100\right) + 10\left(x^4 - 10x^2 + 100\right) \\ &= x^6 - 10x^4 + 100x^2 + 10x^4 - 100x^2 + 1000 \\ &= x^6 + 1000 = 1000 + x^6 \end{aligned}$$

4. $(x+1)^3 - 27$

Solution: We will factor via the difference of cubes theorem, $A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$. In this case, $A = x+1$ and $B = 3$.

$$\begin{aligned}(x+1)^3 - 27 &= (x+1)^3 - 3^3 = ((x+1)-3)\left((x+1)^2 + 3(x+1) + 3^2\right)\\ &= (x-2)\left(x^2 + 2x + 1 + 3x + 3 + 9\right) = (x-2)\left(x^2 + 5x + 13\right)\end{aligned}$$

5. $-2a^7 - 2a^4b^9$

Solution:

$$\begin{aligned}-2a^7 - 2a^4b^9 &= -2a^4\left(a^3 + b^9\right) = -2a^4\left(a^3 + \left(b^3\right)^3\right)\\ &= -2a^4\left(a + b^3\right)\left(a^2 + ab^3 + \left(b^3\right)^2\right)\\ &= -2a^4\left(a + b^3\right)\left(a^2 - ab^3 + b^6\right)\end{aligned}$$

6. $(a+2)^3 + (a-2)^3$

Solution: We will factor via the sum of cubes theorem, $X^3 + Y^3 = (X+Y)\left(X^2 - XY + Y^2\right)$. In this case, $X = a+2$ and $Y = a-2$.

$$\begin{aligned}(a+2)^3 + (a-2)^3 &= ((a+2)+(a-2))\left((a+2)^2 + (a+2)(a-2) + (a-2)^2\right)\\ &= (a+2+a-2)\left(\left(a^2+4a+4\right) - \left(a^2-4\right) + \left(a^2-4a+4\right)\right)\\ &= (2a)\left(a^2 + 4a + 4 - a^2 + 4 + a^2 - 4a + 4\right)\\ &= 2a\left(a^2 + 12\right) = 2a\left(a^2 + 12\right)\end{aligned}$$

7. $a^6 - b^6$

Solution: We start by the difference of squares theorem.

$$a^6 - b^6 = \left(a^3 + b^3\right)\left(a^3 - b^3\right)$$

Now both factors will further factor, via the sum- and difference of cubes theorems. Since

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right) \quad \text{and} \quad a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

we have that

$$\begin{aligned}a^6 - b^6 &= \left(a^3 + b^3\right)\left(a^3 - b^3\right)\\ &= (a+b)\left(a^2 - ab + b^2\right)(a-b)\left(a^2 + ab + b^2\right)\\ &= (a+b)(a-b)\left(a^2 + ab + b^2\right)\left(a^2 - ab + b^2\right)\end{aligned}$$

