Recall the zero product rule.

Theorem: Suppose we multiply several numbers, and the product is zero. Then

1. One of the factors must be zero; and
2. The value of all other factors is irrelevant.

This property is very special – only a product of zero allows us to fixate on one factor while ignoring the other factors. No other number has this property. Because of this property, factoring is an essential tool for solving equations of degree 2, 3, 4, and on.

This is one of the reasons why we find factoring so useful. So far, we have learned to factor out the greatest common factor (or GCF) from an expression. We will now further progress. However, factoring the greatest common factor remains important and it must be the first thing we consider when we factor n expression. Often times, we will make several steps. If factoring out the GCF is needed, it should always be the first thing we do, otherwise we might not apply the other techniques. Now we will see one new factoring technique, the difference of squares theorem.

Consider the expression $2a - 1$. When asked to find the opposite of this expression, a common error is to think that the opposite of $2a - 1$ is $2a + 1$. This is of course incorrect. To get to the opposite, we must multiply by -1 and then apply the distributive law. The opposite of $2a - 1$ is $-2a + 1$.

However, the relationship between $2a - 1$ and $2a + 1$ is important enough for it to have a name.

Definition: Suppose we are given an algebraic expression that is a sum (or a difference). A **conjugate** of the expression is obtained by changing the sign in front of just one term in the expression.

For example, a conjugate of $x + 3$ is $x - 3$. Another possible conjugate of $x + 3$ is $-x + 3$. As long as the expression is organized so that the variable is first, we will be in the habit of changing the second sign. This is not a strict rule however, just a habit.

Conjugates are very useful in algebra. Because of the nearly identical terms and alternating signs, working with conjugates results in cancellations.

Example 1. Consider the expressions $x + 3$ and $x - 3$. Add, subtract, and multiply these expressions. Simplify your answer.

Solution: a) Let us add the conjugates. We drop the parentheses and combine like terms.

$$(x + 3) + (x - 3) = 2x$$

In this case, the first term doubles up, and the second term is canceled out.

b) Now we will add the conjugates. To subtract is to add the opposite.

$$(x + 3) - (x - 3) = x + 3 - x + 3 = 6$$

Now the first term is cancelled out and the second term doubles.

c) Now we multiply the conjugates. We apply FOIL (first, outer, inner last) and then combine like terms, usually O and I from FOIL.

$$(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9$$

Because of the symmetries, when we combine the like terms from O and I, we have complete cancellation. Only conjugates do this.

When we multiply conjugates, O and I in FOIL completely cancel out each other. We are left with the first and last terms only. To express this in more general terms, $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$. This is a fairly reasonable statement. If we start out with conjugates, the symmetries and differences result in such a cancellation. We get a much more suprising statement if we swap the two sides.

Theorem: (The difference of squares theorem) For all quantities a and b,

$$a^2 - b^2 = (a+b)(a-b)$$

In words, the difference of two squares can always be factored into a pair of conjugates.

Example 2. Completely factor each of the following expressions.

a) $3a^2 - 12$ b) $2x^2 - 50y^2$ c) $x^2 - 1$ d) $x^2 + 1$ e) $-x^6 + 49$

Solution: a) Consider the expression $3a^2 - 12$. We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3a^2 - 12 = 3\left(a^2 - 4\right)$$

What is in the parentheses, $a^2 - 4$, can be further factored via the difference of squares theorem.

$$3\left(a^2 - 4\right) = 3\left(a^2 - 2^2\right) = 3(a+2)(a-2)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$3(a+2)(a-2) = 3\left(a^2 - 2a + 2a - 4\right) = 3\left(a^2 - 4\right) = 3a^2 - 12$$

and so our answer, $\boxed{3(a+2)(a-2)}$ is correct.

Notice that $3a^2$ and 12 are not squares. Therefore, if we did not start with the greatest common factor, then we couldn't apply the difference of squares theorem. It is essential that we always start with the greatest common factor.

b) Consider the expression $2x^2 - 50y^2$. We start with the greatest common factor (or GCF). In this case, the GCF is 2. We factor it out:

$$2x^2 - 50y^2 = 2\left(x^2 - 25y^2\right)$$

Now $x^2 - 25y^2$ can be factored further via the difference of squares theorem.

$$2\left(x^2 - 25y^2\right) = 2\left(x^2 - (5y)^2\right) = 2(x+5y)(x-5y)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$2(x+5y)(x-5y) = 2\left(x^2 - 5xy + 5xy - 25\right) = 2\left(x^2 - 25y^2\right) = 2x^2 - 50y^2$$

and so our answer, $\boxed{2(x+5y)(x-5y)}$ is correct.

c) Consider the expression $x^2 - 1$. This is the simplest and possibly the most famous difference of two squares. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 1$ can be factored via the difference of squares theorem.

$$x^2 - 1 = x^2 - 1^2 = (x+1)(x-1)$$

The expressions in neither parentheses can not be further factored and so we are done. We check our work by multiplication:

$$(x+1)(x-1) = x^2 - x + x - 1 = x^2 - 1$$

and so our answer, $\boxed{(x+1)(x-1)}$ is correct.

It is a common mistake to cofuse $x^2 - 1$ with $(x-1)^2$. The expression $(x-1)^2$ is a complete square in which O and I from FOIL are identical, so they double up. Only conjugates cause cancellation of the $x-$terms.

$$\begin{aligned}(x-1)^2 &= (x-1)(x-1) = x^2 - x - x + 1 = x^2 - 2x + 1 \\ x^2 - 1 &= (x+1)(x-1)\end{aligned}$$

d) Consider the expression $x^2 + 1$. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. In addition, $x^2 + 1$ can NOT be factored via the difference of squares theorem. **The sum of two squares can never be factored.** So, there is nothing that can be done here, and the final answer is $\boxed{x^2 + 1}$.

e) Conside the expression $-x^6 + 49$. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. Before we proceed any further, we rearrange the terms so that the difference of squares becomes easier to observe.

$$-x^6 + 49 = 49 - x^6$$

This factors via the difference of sqaures theorem. It is x^3 that we need to square to obtain x^6.

$$49 - x^6 = 7^2 - \left(x^3\right)^2 = \left(7 + x^3\right)\left(7 - x^3\right)$$

What is in both parentheses, $7 + x^3$ and $7 - x^3$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$\left(7 + x^3\right)\left(7 - x^3\right) = 49 - 7x^3 + 7x^3 - x^6 = 49 - x^6$$

and so our answer, $\boxed{\left(7 + x^3\right)\left(7 - x^3\right)}$ is correct.

Please note that there is another method to solve this problem that might be more strategic. When dealing with algebraic expressions, we prefer to have the unknown first, in descending order of degrees, and then the number. Another option we have is to factor out -1 rigth away. This way we are dealing with a much more familiar situation

$$49 - x^6 = -x^6 + 49 = -1\left(x^6 - 49\right) = -\left(\left(x^3\right)^2 - 7^2\right) = -\left(x^3 + 7\right)\left(x^3 - 7\right)$$

also perfectly correct.

Example 3. Completely factor the expression $2p^4 - 162$.

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
2p^4 - 162 &= 2\left(p^4 - 81\right) && \text{re-write both quantities as squares}\\
&= 2\left(\left(p^2\right)^2 - 9^2\right) && \text{factor via the difference of squares theorem}\\
&= 2\left(p^2+9\right)\left(p^2-9\right) && \text{the second factor will factor again, the first will not!}\\
&= 2\left(p^2+9\right)\left(p^2-3^2\right) && \text{factor via the difference of squares theorem}\\
&= 2\left(p^2+9\right)(p+3)(p-3)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2-3p+3p-9\right) = 2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}}\\
&= 2\left(p^4-9p^2+9p^2-81\right) = 2\left(p^4-81\right) = 2p^4-162
\end{aligned}$$

Thus our solution, $\boxed{2\left(p^2+9\right)(p+3)(p-3)}$ is correct.

Example 4. Solve each of the following equations. Make sure to check your solution.

a) $x^2 = 9$ b) $x^4 = 9x^3$ c) $8x^3 = 50x^2$ d) $8p^3 = 50p$

Solution: a) Since the equation $x^2 = 9$ is of a higher degree than one, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned}
x^2 &= 9 && \text{subtract } 9\\
x^2 - 9 &= 0 && \text{factor via the difference of squares theorem}\\
x^2 - 3^2 &= 0\\
(x+3)(x-3) &= 0
\end{aligned}$$

A product can only be zero if one of its factors is zero. $(x+3)(x-3) = 0$ means that either $x - 3 = 0$ or $x + 3 = 0$. We solve these linear equations separately and obtain $\boxed{3 \text{ and } -3}$. We check: $3^2 = 9$ and $(-3)^2 = 9$.

Note: one could ask why the four steps if we could just conclude from $x^2 = 9$ that then $x = \pm 3$. This shortcut, called the square root property, is perfectly fine, as long as we remember that there are *two* numbers whose square is 9: the numbers 3 and -3. It is a common but serious error to go from $x^2 = 9$ to $x = 3$. One advantage of the difference of squares theorem is that it will not allow us to forget about the negative solution.

b) The equation $x^4 = 9x^3$ is of a higher degree than one. This is indeed a degree four equation. Therefore, our only method to solve it is to reduce one side to zero, factor, and then apply the zero product rule. It is preferred not to create a negative leading coefficient, and so we will subtract $9x^3$ from both sides.

$$\begin{aligned}
x^4 &= 9x^3 && \text{subtract } 9x^3\\
x^4 - 9x^3 &= 0 && \text{factor out the GCF}\\
x^3(x-9) &= 0
\end{aligned}$$

We apply the zero property. $x^3(x-9) = 0$ or $x \cdot x \cdot x \cdot (x-9) = 0$ means that either $x = 0$ or $x - 9 = 0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2 = 9 \cdot 0$ and $9^2 = 9 \cdot 9$ and so our solution is correct.

c) The equation $8x^3 = 50x^2$ is of a higher degree than one, so our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then LHS $= 8 \cdot 0^3 = 8 \cdot 0 = 0$ and RHS $= 50 \cdot 0^2 = 50 \cdot 0 = 0$. And if $x = \frac{25}{4}$, then

$$\text{LHS} = 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \quad \text{and} \quad \text{RHS} = 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8}$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

d) Since the equation $8p^3 = 50p$ is of a degree higher than one, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2 - 25\right) &= 0 \\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve each equation for p.

$$\begin{aligned} 2p + 5 &= 0 && \text{or} && 2p - 5 = 0 && \text{or} && 2p = 0 \\ 2p &= -5 && \text{or} && 2p = 5 && \text{or} && p = 0 \\ p &= -\frac{5}{2} && \text{or} && p = \frac{5}{2} \end{aligned}$$

We check all three solutions. If $p = -\frac{5}{2}$, then

$$\text{LHS} = 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \text{ and } \text{RHS} = 50\left(-\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125$$

If $p = \frac{5}{2}$, then

$$\text{LHS} = 8\left(\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{125}{8} = 125 \quad \text{and} \quad \text{RHS} = 50\left(\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{5}{2} = \frac{250}{2} = 125$$

and if $p = 0$, then

$$\text{LHS} = 8 \cdot 0^3 = 8 \cdot 0 = 0 \quad \text{and} \quad \text{RHS} = 50 \cdot 0 = 0$$

Thus all three solutions, $\boxed{-\frac{5}{2},\ 0,\ \text{and}\ \frac{5}{2}}$ are correct.

Example 5. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote this number by x. Then the equation is simply

$$x^2 = x$$

This is a quadratic equation, so we will reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

We solve the linear equations $x = 0$ and $x - 1 = 0$ and obtain the solutions 0 and 1. Clearly both of these numbers square equals to the number. What is more important, we proved that no other number has this property.

Example 6. Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Solution: Let us denote the number by x. The equation is then $x^3 = 4x$. We solve this equation.

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3 - 4x &= 0 && \text{factor out the GCF} \quad (\text{it is } x) \\ x\left(x^2 - 4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{array}{ccccc} x = 0 & \text{or} & x + 2 = 0 & \text{or} & x - 2 = 0 \\ x = 0 & \text{or} & x = -2 & \text{or} & x = 2 \end{array}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: $\boxed{0,\ 2,\ \text{and}\ -2.}$

Practice Problems

1. Factor out the greatest common factor from each of the following.

   a) $10a^2b^2 - 15ab^3 + 25a^2b^3c$ c) $a^2 - a^3 + a^4$ e) $x^5 - 2x^4 + 4x^3$

   b) $6x^3 - 3x^2 - 15x^4$ d) $6a^2b + 12a^3b - 30a^3b^2$ f) $3xy\,(a-3) + 8t\,(a-3) - 200x^5\,(a-3)$

2. Factor out -1 from each of the following.

   a) $x^3 - x^5 + 2$ b) $-x^2 + 3x - 1$ c) $-x^2 + 3x - 5$

3. Factor each of the following via the difference of squares theorem.

   a) $x^2 - 49$ b) $9a^2 - 25$ c) $x^2 - 1$ d) $y^6 - 100$

4. Completely factor each of the following.

   a) $5a^2 - 45$ e) $x^3 - x$ i) $a^2 - (x-1)^2$ m) $-2x^4 + 162$

   b) $2m^4 - 2n^4$ f) $5x^3y^4 - 80x^3$ j) $-16 + a^4$ n) $5a^3b^2 - 15ab$

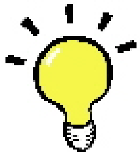
   c) $2x^4 - 8x^2$ g) $a^2\,(x-1) - 9\,(x-1)$ k) $600ab^2 - 6ab^4$

   d) $3a - 12ab^2$ h) $18a^2x^2 - 50x^2$ l) $36x^2y^3 + 4x^4y^3$

5. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1) = 0$ c) $2\,(x-2)(x+3) = 0$ e) $x^2 + 6x = 0$ g) $3x^3 = 75x$

   b) $x\,(x-2)(x+3) = 0$ d) $x^2 = 4$ f) $3x^3 = 75x^2$ h) $45a^4 = 20a^2$



Answers

Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2(2x-5x^2-1)$ c) $a^2(a^2-a+1)$ d) $6a^2b(2a-5ab+1)$
   e) $x^3(x^2-2x+4)$ f) $(a-3)(3xy+8t-200x^5)$
2. a) $-(-x^3+x^5-2)$ b) $-(x^2-3x+1)$ c) $-(x^2-3x+5)$
3. a) $(x+7)(x-7)$ b) $(3a+5)(3a-5)$ c) $(x+1)(x-1)$ d) $(y^3+10)(y^3-10)$
4. a) $5(a+3)(a-3)$ b) $-2(n-m)(m+n)(m^2+n^2)$ c) $2x^2(x+2)(x-2)$ d) $-3a(2b+1)(2b-1)$
   e) $x(x+1)(x-1)$ f) $5x^3(y-2)(y+2)(y^2+4)$ g) $(a-3)(a+3)(x-1)$ h) $2x^2(3a-5)(3a+5)$
   i) $(a+x-1)(a-x+1)$ j) $(a-2)(a+2)(a^2+4)$ k) $-6ab^2(b-10)(b+10)$ l) $4x^2y^3(x^2+9)$
   m) $-2(x^2+9)(x+3)(x-3)$ n) $5ab(a^2b-3)$
5. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $2, -2$ e) $0, -6$ f) $0, 25$ g) $-5, 0, 5$ h) $-\frac{2}{3}, 0, \frac{2}{3}$

