For all polynomials, factoring is unique. For example, the expression x^2-16 can *only* be factored as $(x+4)(x-4)$. As long as we insist to completely factor a polynomial, there is just one correct form. This is different from equations that can have more than one solution.

Because of its uniqueness, we are not forced to develop mathods to systematically find all factored form; if we found one, we found *it*. Because of this, we are allowed to use trial and error to 'stumble into' the factored form. **Trial and error** (or by inspection) refers to a factoring method where we make educated guesses that allow us to quickly factor a quadratic expression.

In what follows, we will only focus on expressions in which the coefficient of the quadraic term is 1. Suppose we expand the expression $(x+a)(x+b)$.

$$(x+a)(x+b)=x^2+bx+ax+ab=x^2+(b+a)x+ab$$

The result is $x^2+(a+b)x+ab$. Notice that the linear coefficient is the sum of a and b, and the number term is the product of a and b. We can use these facts to quickly factor a quadratc expression such as $x^2+7x+12$. We will start with the easiest case: when all signs are +.

Case 1: All signs are + in the expression to be factored.

Example 1. Completely factor the expression $x^2+7x+12$.

Solution: If $x^2+7x+12$ is factored into $(x+a)(x+b)$, then factoring is just a matter of finding a and b. Consider the equation $a+b=7$. This equation has infinitely many solutions. For any value of a there is a value of b that works. If $a=1$, then $b=6$, if $a=10$, then $b=-3$, and so on. This is not the case with the equation $ab=12$. As long as we are looking for integer values, there is a finite list of how the product of two numbers is 12. Because of this, we will **always start with the number term** that is the product of a and b.

We can quickly list all the pairs of positive numbers with a product of 12.

12	
1	12
2	6
3	4

Now we consider the three pairs as candidates for a and b. We are looking for the pair with sum 7. Clearly that is 3 and 4. Once we found a and b with product 12 and sum 7, we have the factored form: $x^2+7x+12=\boxed{(x+3)(x+4)}$

Naturally, things are not always as simple. Consider now the second case, when the last term is positive but the second term is negative.

Case 2: The third sign is + in the expression to be factored, and the second sign is negative.

Example 2. Completely factor the expression $x^2-17x+30$.

Solution: We are looking for two numbers a and b with a product of 30 and a sum of -17. A positive product indicates that a and b are either both positive or both negative. Because of the negative second sign, both positive is impossible. Therefore, we are looking for two negative numbers.

As always, we start with the equation $ab=30$. We list all the pairs of negative numbers with a product of 30.

30	
-1	-30
-2	-15
-3	-10
-5	-6

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum -17. Clearly that is -2 and -15. Once we found a and b with product 30 and sum -17, we have the factored form: $x^2 - 17x + 30 = \boxed{(x-2)(x-15)}$

Case 3: The third sign is - in the expression to be factored.

Example 3. Completely factor the expression $x^2 - 2x - 48$.

Solution: We are looking for two numbers a and b with a product of -48 and a sum of -2. A negative product indicates that one of a and b is positive and the other is negative. Now we inspect the second sign. If the sum of a positive and a negative number is negative, then between the two of them, the negative one has the greater absolute value. These observations will make our task much easier.

As always, we start with the equation $ab = -48$. We list all the pairs of positive numbers with a product of 48, and put a $-$ sign in front of the greater one.

-48	
1	-48
2	-24
3	-16
4	-12
6	-8

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum -2. Clearly that is -8 and 6. Once we found a and b with product -48 and sum -2, we have the factored form: $x^2 - 2x - 48 = \boxed{(x-8)(x+6)}$

In the next example, the second term is positive.

Example 4. Completely factor the expression $x^2 + 11x - 60$.

Solution: We are looking for two numbers a and b with a product of -60 and a sum 11. A negative product indicates that one of a and b is positive and the other is negative. Now we inspect the second sign. If the sum of a positive and a negative number is positive, then between the two of them, the negative one has the smaller absolute value. These observations will make our task much easier.

We start with the equation $ab = -60$. We list all the pairs of positive numbers with a product of 60, and put a $-$ sign in front of the smaller one.

-60	
-1	60
-2	30
-3	20
-4	15
-5	12
-6	10

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum 11. Clearly that is -4 and 15. Therefore, $x^2 + 11x - 60 = \boxed{(x+15)(x-4)}$

We can check by mutiplication: $(x+15)(x-4) = x^2 - 4x + 15x - 60 = x^2 + 11x - 60$, so our solution is correct.

This method is quick and easy. However, it only works for simple expressions that start with x^2. Consider for example the product $(2x+3)(x+5) = 2x^2 + 10x + 3x + 15 = 2x^2 + 13x + 15$. The middle term is clearly not the sum of 3 and 5. Because of uniqueness of factoring, trial and error is still a good approach, but the middle term is no longer just the sum.

Example 5. Solve the equation $(x-2)(x-4)=24$

Solution: We might be tempted to use the factored form on the left-hand side, but it cannot be used because the other side is not zero. So we need to expand the product, reduce one side to zero and then factor.

$$\begin{aligned} (x-2)(x-4) &= 24 && \text{expand product} \\ x^2-6x+8 &= 24 && \text{subtract } 24 \\ x^2-6x-16 &= 0 \end{aligned}$$

To factor $x^2-6x-16$, we need to find two integers a and b with product -16 and sum -6. The negative sign in -16 indicates that one is positive, the other one is negative. The negative sign in -6 indicates that the negative number has the greater absolute value.

-16	
1	-16
2	-8
4	-4

The only pair with sum -6 is -8 and 2. Therefore, $x^2-6x-16=(x-8)(x+2)$. Applying the zero product rule, we obtain $x=8$ and $x=-2$.

We check: if $x=8$, then LHS $=(8-2)(8-4)=6\cdot 4=24=$ RHS ✓

and if $x=-2$, then LHS $=(-2-2)(-2-4)=-4(-6)=24=$ RHS ✓

So both $\boxed{8 \text{ and } -2}$ work.

If there is a leading coefficient, this method only works if it is also the greatest common factor and can be factored out.

Example 6. One side of a rectangle is 6 feet shorter than twice another side. Find the sides of the rectangle if we also know that its area is $140\,\text{ft}^2$.

Solution: If we label one side by x, the other side is $2x-6$. The equation will express the area of the rectangle.

$$\begin{aligned} x(2x-6) &= 140 && \text{distribute } x \\ 2x^2-6x &= 140 && \text{subtract } 140 \\ 2x^2-6x-140 &= 0 && \text{factor out } 2 \\ 2\left(x^2-3x-70\right) &= 0 \end{aligned}$$

We will factor $x^2-3x-70$. We are looking for two integers with product -70 and sum -3. These are easily found: -10 and 7. Therefore, $x^2-3x-70=(x-10)(x+7)$. Back to the equation:

$$2(x-10)(x+7)=0 \quad \Longrightarrow \quad x_1=10 \ \text{ and } \ x_2=-7$$

The two solutions of this equation are 10 and -7. Since we are looking for a distance and distances cannot be negative, -7 is easily ruled out. If the shorter side is x, then the other side is $2x-6=2\cdot 10-6=14$. So the two sides are $\boxed{10\,\text{ft and } 14\,\text{ft}}$ long.

Example 7. Find all numbers that are exactly six less than their own square.

Solution: If we label such a number by x, then the equation will be $x^2 = x - 6$.

$$\begin{aligned} x^2 &= x - 6 \qquad \text{subtract } x \text{ and add } 6 \\ x^2 - x - 6 &= 0 \end{aligned}$$

We quickly find -3 and 2 as two numbers with sum -1 and product -6.

$$(x-3)(x+2) = 0 \quad \Longrightarrow \quad x_1 = 3 \text{ and } x_2 = -2$$

We check: 3 is indeed 6 less than 9, and -2 is indeed less than 4. So our asnwer is $\boxed{-2 \text{ and } 3}$. Perhaps even more importantly, we also proved that there is no other number with this property.

Practice Problems

1. Completely factor each of the following using the trial and error method.

a) $x^2 + 2x - 15$
b) $x^2 - 12x + 32$
c) $x^2 - 2x - 3$
d) $x^2 - 10x + 25$
e) $x^2 + 9x + 20$
f) $x^2 - x - 20$
g) $x^2 - 5x + 6$
h) $x^2 - 5x - 6$
i) $2x^2 - 8x - 42$
j) $-3x^2 - 3x + 6$

2. Solve each of the following equations. Make sure to check your solutions.

a) $(w+5)(w-1) = 0$
b) $(w+5)(w-1) = 55$
c) $(x-2)(x+3) = 50$
d) $5x^3 = 10x^2 + 75x$
e) $(2x-1)^2 - x = 3x(x-1)$
f) $(x+5)^2 + (x-1)^2 = (x+6)^2 + 2$

3. Find all numbers satisfying the given conditions.

a) The square of the number is twenty greater than the number.

b) The sum of the square of the number and three times the number is 70.

4. a) One side of a rectangle is twelve feet shorter than three times another side. Find the sides of this rectangle if we also know that the area of this rectangle is $420\,\text{ft}^2$.

b) One side of a rectangle is twelve feet longer than three times another side. Find the sides of this rectangle if we also know that the area of this rectangle is $288\,\text{ft}^2$.

Answers

Discussion:

Practice Problems

1. a) $(x-3)(x+5)$ b) $(x-4)(x-8)$ c) $(x+1)(x-3)$ d) $(x-5)^2$ e) $(x+4)(x+5)$
   f) $(x+4)(x-5)$ g) $(x-2)(x-3)$ h) $(x-6)(x+1)$ i) $2(x-7)(x+3)$ j) $-3(x+2)(x-1)$
2. a) $-5, 1$ b) $6, -10$ c) $7, -8$ d) $-3, 0, 5$ e) 1 f) $6, -2$ 3. a) $5, -4$ b) $-10, 7$
4. a) $14\,\mathrm{ft}$ by $30\,\mathrm{ft}$ b) $8\,\mathrm{ft}$ by $36\,\mathrm{ft}$

