Part 1 - Definitions

Some sets are small enough for us to list their elements. With larger sets, we develop notation that helps in defining the set without having to write too much. Consider the following two sets.

$$A = \{x \in \mathbb{Z} : x > 2 \text{ and } x < 7\} \qquad \text{and} \qquad B = \{x \in \mathbb{R} : x > 2 \text{ and } x < 7\}$$

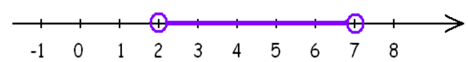
Although the definitions appear to be similar, set A is much smaller than set B. Set A contains all integers greater than 2 and less than 7. That is, A is simply the set $\{3, 4, 5, 6\}$ containing only four elements.

Set B has many more elements, because it is the set of all *real numbers* greater than 2 and less than 7. That means that B contains numbers such as $2,5$, 2.01, 3.1, 6.999999998. If we think of the numbers 2.1, 2.01, 2.001, 2.0001, and so on, these are already infinitely many numbers in B.

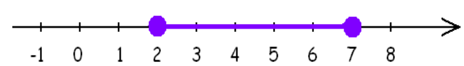
How could we describe set B with less writing? One option is to write the compound inequality $x > 2$ and $x < 7$ in a more effective form, as $2 < x < 7$. Still, we should be able to do better than $B = \{x \in \mathbb{R} : 2 < x < 7\}$.

The set of all real numbers x with $2 < x < 7$ can be also expressed using interval notation.

Definition: The set $\{x \in \mathbb{R} : 2 < x < 7\}$ is also called an **interval**, and is denoted by $(2, 7)$. The endpoints of the interval, 2 and 7 are not elements of the set. Such an interval is called an **open interval**.

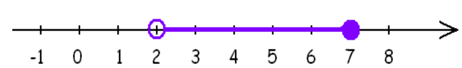


Definition: The set $\{x \in \mathbb{R} : 2 \leq x \leq 7\}$ is denoted by $[2, 7]$. The endpoints of the interval, 2 and 7 belong to the set. Such an interval is called a **closed interval**.

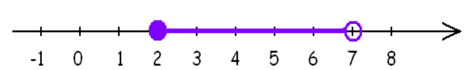


We will see in future courses that open and closed intervals have very different properties. For example, the closed interval $[2, 7]$ has a smallest and greatest element. At the same time, the open interval $(2, 7)$ has no smallest and greatest element. With respect to the endpoints, there are two other possibilities, as follows.

Definition: The set $\{x \in \mathbb{R} : 2 < x \leq 7\}$ is denoted by $(2, 7]$.



Definition: The set $\{x \in \mathbb{R} : 2 \leq x < 7\}$ is denoted by $[2, 7)$.



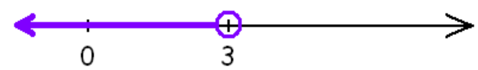
We also often use interval notation when presenting solution sets of inequalities. Interval notation can also be applied to express simple sets that can be obtained from inequalities such as $x < 3$, $x \leq 3$, $x > 3$, or $x \geq 3$.

Definition: The set $\{x \in \mathbb{R} : x > 3\}$ in interval notation is denoted by $(3, \infty)$.

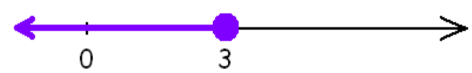
Definition: The set $\{x \in \mathbb{R} : x \geq 3\}$ in interval notation is denoted by $[3, \infty)$.

Notice that in case of both $x > 3$ and $x \geq 3$, the closing parentheses indicate that ∞ does not belong to the set. We can also consider the set $(3, \infty)$ to be an open interval. For example, this set has neither smallest nor greatest element.

Definition: The set $\{x \in \mathbb{R} : x < 3\}$ in interval notation is denoted by $(-\infty, 3)$.



Definition: The set $\{x \in \mathbb{R} : x \leq 3\}$ in interval notation is denoted by $(-\infty, 3]$.



Part 2 - Operations

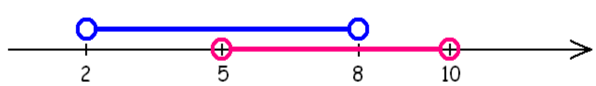
Intervals are sets, and so we can perform set operations on them.

Example 1. Perform each of the following set operations on the intervals.

a) $(2,8) \cap (5,10)$ c) $[1,4] \cap (2,5)$ e) $[-1,2) \cap (3,6]$ g) $(2,8) \cap [4,7]$

b) $(2,8) \cup (5,10)$ d) $[1,4] \cup (2,5)$ f) $[-1,2) \cup (3,6]$ h) $(2,8) \cup [4,7]$

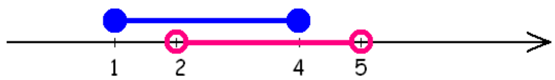
Solution: a) Plotting the two intervals on the same number line is extremely helpful. We will use the same picture for taking unions and intersections.



The intersection of the intervals $(2,8)$ and $(5,10)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 5 and 8. We consider the endpoints: 5 is not in both sets because 5 is not in $(5,10)$. Similarly, 8 is not in both sets because it is not in $(2,8)$. Consequently, the intersection of the two intervals is $\boxed{(5,8)}$.

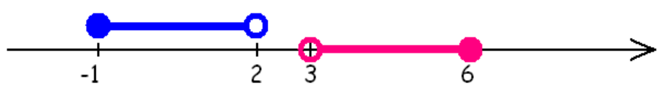
b) The union of the intervals $(2,8)$ and $(5,10)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 2 and 10. We consider the endpoints: 2 is not in either set, so it is not in the union. Similarly, 10 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $\boxed{(2,10)}$.

c) The intersection of the intervals $[1,4]$ and $(2,5)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 2 and 4. We consider the endpoints: 2 is not in both sets because 2 is not in $(2,5)$. However, 4 is in both sets. Therefore, the intersection of the two intervals is $\boxed{(2,4]}$.



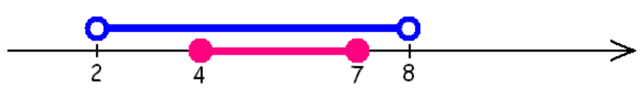
d) The union of the intervals $[1,4]$ and $(2,5)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 1 and 5. We consider the endpoints: 1 is in $[1,4]$, so it is in the union. On the other hand, 5 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $\boxed{[1,5)}$.

e) The intersection of the intervals $[-1,2)$ and $(3,6]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\boxed{\varnothing}$.



f) The union of the intervals $[-1,2)$ and $(3,6]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $\boxed{[-1,2)\cup(3,6]}$.

g) After we plotted the picture, we might notice that one interval is a subset of the other. In light of that, the answers will not be surprising. (Recall that if A and B are sets and $A\subseteq B$, then $A\cup B=B$ and $A\cap B=A$). The intersection of the intervals $(2,8)$ and $[4,7]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. The answer is $\boxed{[4,7]}$.



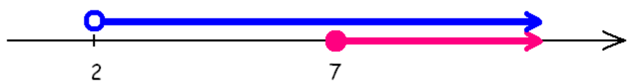
h) The union of the intervals $(2,8)$ and $[4,7]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union is $\boxed{(2,8)}$.

Example 2. Perform each of the following set operations on the intervals.

a) $(2, \infty) \cap [7, \infty)$ c) $[0, \infty) \cap (-\infty, 1)$ e) $(-\infty, 4] \cap [9, \infty)$

b) $(2, \infty) \cup [7, \infty)$ d) $[0, \infty) \cup (-\infty, 1)$ f) $(-\infty, 4] \cup [9, \infty)$

Solution: a) Notice again that one interval is a subset of the other. The intersection of the intervals $(2, \infty)$ and $[7, \infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, the intersection is $\boxed{[7, \infty)}$.

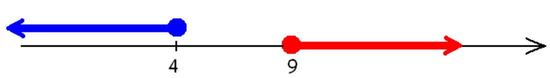


b) The union of the intervals $(2, \infty)$ and $[7, \infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is $\boxed{(2, \infty)}$.

c) The intersection of the intervals $[0, \infty)$ and $(-\infty, 1)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 0 and 1. We consider the endpoints: 0 is in both sets but 1 is not, since 1 is not in $(-\infty, 1)$.Therefore, the intersection of the two intervals is $\boxed{[0, 1)}$.

d) The union of the intervals $[0, \infty)$ and $(-\infty, 1)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, this is the entire number line. Consequently, the union of the two intervals is $\mathbb{R}$, the set of all real numbers. This can be also expressed using interval notation, as $\boxed{(-\infty, \infty)}$.

e) The intersection of the intervals $(-\infty, 4]$ and $[9, \infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\boxed{\varnothing}$.



f) The union of the intervals $(-\infty, 4]$ and $[9, \infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $\boxed{(-\infty, 4] \cup [9, \infty)}$.

Practice Problems

1. Re-write each of the given sets using interval notation.

a) $\{x \in \mathbb{R} : x \geq 3\}$
b) $\{m \in \mathbb{R} : -2 < m \leq 9\}$
c) $\mathbb{R}$
d) $\{y \in \mathbb{R} : y < -5 \text{ or } y > 4\}$
e) $\{x \in \mathbb{R} : x < 12\}$
f) $\{r \in \mathbb{R} : r < 10 \text{ and } r \geq 6\}$
g) $\{x \in \mathbb{R} : x < -2 \text{ or } x > 2\}$
h) $\{t \in \mathbb{R} : t > 0 \text{ and } t < 7\}$
i) $\{a \in \mathbb{R} : 3 \leq a \leq 4\}$

2. Perform each of the set operations on the intervals.

a) $(1,5) \cup (2,7)$
b) $(1,5) \cap (2,7)$
c) $[1,5] \cup [2,7]$
d) $[1,5] \cap [2,7]$
e) $[1,5] \cup (2,7)$
f) $[1,5] \cap (2,7)$
g) $(3,8) \cup (-1,10)$
h) $(3,8) \cap (-1,10)$
i) $[3,8] \cup [-1,10]$
j) $[3,8] \cap [-1,10]$
k) $[3,8] \cup (-1,10)$
l) $[3,8] \cap (-1,10)$
m) $[-2,2] \cup (4,7)$
n) $[-2,2] \cap (4,7)$
o) $[-2,1) \cup (0,8]$
p) $[-2,1) \cap (0,8]$
q) $[5,10) \cup [7,11)$
r) $[5,10) \cap [7,11)$

3. Perform each of the set operations on the intervals.

a) $(-\infty,4) \cup (-\infty,8)$
b) $(-\infty,4) \cap (-\infty,8)$
c) $(-\infty,4] \cup (-\infty,8]$
d) $(-\infty,4] \cap (-\infty,8]$
e) $(-\infty,4] \cup (-\infty,8)$
f) $(-\infty,4] \cap (-\infty,8)$
g) $(-\infty,5) \cup (3,\infty)$
h) $(-\infty,5) \cap (3,\infty)$
i) $(-\infty,5] \cup [3,\infty)$
j) $(-\infty,5] \cap [3,\infty)$
k) $(-\infty,5] \cup (3,\infty)$
l) $(-\infty,5] \cap (3,\infty)$
m) $(-\infty,-2) \cup (1,\infty)$
n) $(-\infty,-2) \cap (1,\infty)$
o) $(-\infty,-2] \cup [1,\infty)$
p) $(-\infty,-2] \cap [1,\infty)$
q) $(-\infty,-2] \cup (1,\infty)$
r) $(-\infty,-2] \cap (1,\infty)$

Answers

1. a) $[3, \infty)$ b) $(-2, 9]$ c) $(-\infty, \infty)$ d) $(-\infty, -5) \cup (4, \infty)$ e) $(-\infty, 12)$ f) $[6, 10)$ g) $(-\infty, -2) \cup (2, \infty)$ h) $(0, 7)$ i) $[3, 4]$

2. a) $(1, 7)$ b) $(2, 5)$ c) $[1, 7]$ d) $[2, 5]$ e) $[1, 7)$ f) $(2, 5]$ g) $(-1, 10)$ h) $(3, 8)$ i) $[-1, 10]$ j) $[3, 8]$ k) $(-1, 10)$ l) $[3, 8]$ m) $[-2, 2] \cup (4, 7)$ n) $\varnothing$ o) $[-2, 8]$ p) $(0, 1)$ q) $[5, 11)$ r) $[7, 10)$

3. a) $(-\infty, 8)$ b) $(-\infty, 4)$ c) $(-\infty, 8]$ d) $(-\infty, 4]$ e) $(-\infty, 8)$ f) $(-\infty, 4]$ g) $\mathbb{R} = (-\infty, \infty)$ h) $(3, 5)$ i) $\mathbb{R} = (-\infty, \infty)$ j) $[3, 5]$ k) $\mathbb{R} = (-\infty, \infty)$ l) $(3, 5]$ m) $(-\infty, -2) \cup (1, \infty)$ n) $\varnothing$ o) $(-\infty, -2] \cup [1, \infty)$ p) $\varnothing$ q) $(-\infty, -2] \cup (1, \infty)$ r) $\varnothing$

