Let us first recall a definition.

Definition: Equations that are in x, or in y, or in x and y can be graphed. **The graph of such an equation is the set of all points $P(x, y)$ for which the coordinates x and y form a solution of the equation.**

The slope of a line is a very important concept.

Definition: The slope of a line (usually denoted by m) is a signed ratio expressing the steepness of the line. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the slope of the line connecting these points is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is called the **slope formula**.

Example 1. In each case, find the slope of the line determined by the two points given.

a) $(5, -1)$ and $(1, 3)$ b) $(2, 1)$ and $(6, 3)$ c) $(8, -1)$ and $(-1, -1)$ d) $(7, 3)$ and $(7, -4)$

Solution: a) We find the slope determined by the points. $(5, -1) = (x_1, y_1)$ and $(1, 3) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{1 - 5} = \frac{4}{-4} = -1$$

So the slope is -1.

b) We find the slope determined by the points. $(2, 1) = (x_1, y_1)$ and $(6, 3) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{6 - 2} = \frac{2}{4} = \frac{1}{2}$$

So the slope is $\frac{1}{2}$.

c) We find the slope determined by the points. $(8, -1) = (x_1, y_1)$ and $(-1, -1) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-1)}{-1 - 8} = \frac{0}{-9} = 0$$

So the slope is 0. Note that **the slope of all horizontal lines is zero**.

d) We find the slope determined by the points. $(7, 3) = (x_1, y_1)$ and $(7, -4) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 3}{7 - 7} = \frac{-7}{0} = \text{undefined}$$

So this line has no slope. Note that **vertical lines have no slope**.

Discussion

Suppose we are finding the slope of a line connecting $A(-5, 2)$ and $B(1, 10)$. Ann is using point A as (x_1, y_1) and point B as (x_2, y_2). Beth is using point B as (x_1, y_1) and point A as (x_2, y_2). Compute the slope both ways. Do we get the same answer? Why or why not?

Theorem: Suppose that the equation of a line is $y = mx + b$. Then the slope of this line is m, the coefficient of x in the equation.

For example, the line $y = 2x - 3$ has slope $m = 2$, the line $y = -\frac{2}{3}x + 1$ has slope $m = -\frac{2}{3}$.

Caution! If the line's equation is not in this form, the coefficient of x is NOT the slope. For example, the slope of the line $2x - 3y = -12$ is NOT 2.

The equation $y = mx + b$ is called the **slope-intercept form** of the line.

Example 2. In each case, find the slope of the line given.

a) $y = -3x + 8$ b) $y = \frac{3}{4}x - 2$ c) $y = -2$ d) $x = 7$

Solution: a) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x, -3. So $m = -3$.

b) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x, $\frac{3}{4}$. So $m = \frac{3}{4}$.

c) This equation is horizontal, so its slope is zero. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(1, -2)$ and $(5, -2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-2)}{5 - 1} = \frac{0}{4} = 0$$

Antoher way is to think of the equation $y = -2$ as $y = 0x - 2$ and then the coefficient of x is zero.

d) This equation is vertical, so it does not have a slope. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(7, 1)$ and $(7, 2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{7 - 7} = \frac{0}{0} = \text{undefined}$$

So this line has no slope. (Antoher way is to think of the equation $x = 7$ as an equation in which we can not solve for y, because it doesn't even appear in the equation. Or, $x = 0y + 7$ and then solving for y would result in having to divide by zero.)

The slope-intercept form of a line's equation enables us to plot a line quickly and with very little computation. Every non-vertical line has a slope-intercept form. In case of an integer m, we can think of m as a fraction: $\frac{m}{1}$. If m is already a fraction, great. If $m = \frac{p}{q}$ where p and q are integers, then from one nice lattice point to another, we will make q steps to the right, and p steps up (if p is positive) or down (if p is negative).

$$m = 4 = \frac{4}{1} \implies \text{1 step to the right, 4 steps up}$$

$$m = -1 = \frac{-1}{1} \implies \text{1 step to the right, 1 step down}$$

$$m = \frac{2}{3} \implies \text{3 steps to the right, 2 steps up}$$

$$m = -\frac{5}{6} = \frac{-5}{6} \implies \text{6 steps to the right, 5 steps down}$$

Example 3. Graph the line $y = 2x - 4$

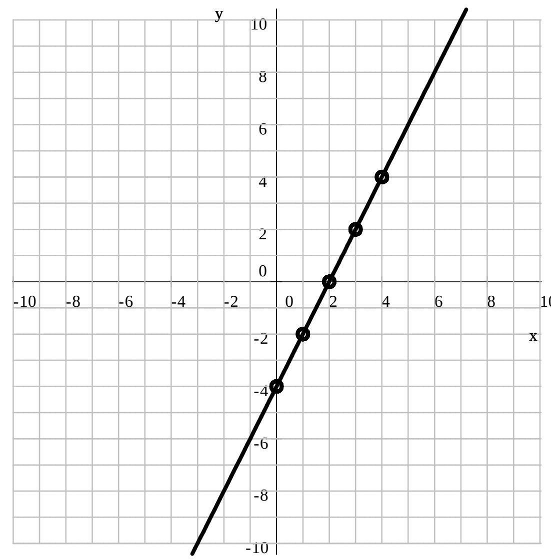
Solution: This line is given in its slope-intercept form. The slope of this line is the coefficient of x. It is $m = 2$.

We graph the $y-$intercept of the line. We obtain the $y-$intercept by substituting $x = 0$ into the equation. Clearly, the $y-$intercept is $(0, -4)$. In general, the $y-$intercept of the line $y = mx + b$ is $(0, b)$. We graph that point first.

We graph additional points using the slope of the line, $m = 2$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$2 = \frac{2}{1} \implies \frac{2 \text{ up}}{1 \text{ to the right}}$$

We start at the $y-$intercept, $(0, -4)$ and to plot additional points, we step 1 to the right, 2 up. We repeat this process several times to plot enough points.



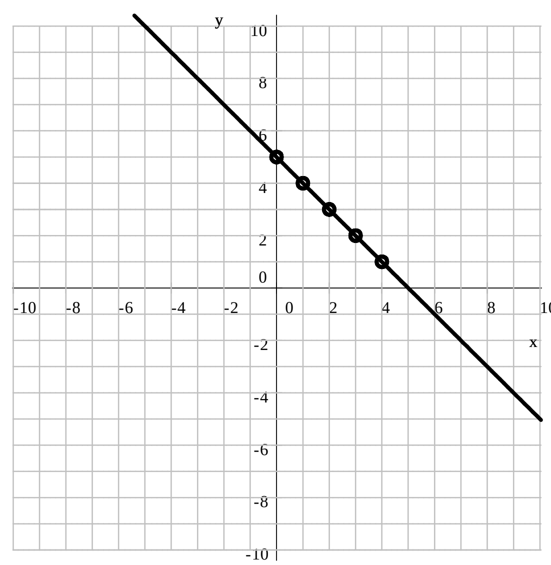
Example 4. Graph the line $y = -x + 5$.

Solution: The slope of this line is $m = -1$, and the $y-$intercept of it is $(0, 5)$. We graph the $y-$intercept first and then graph additional points using the slope of the line.

The slope of this line is $m = -1$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$-1 = \frac{-1}{1} \implies \frac{1 \text{ down}}{1 \text{ to the right}}$$

We start at the $y-$intercept, $(0, 5)$ and to plot additional points, we step 1 to the right, 1 down. We repeat this process several times to plot enough points.



Example 5. Graph the line $x + 2y = 4$

Solution: Step 1. We bring the equation to its slope-intercept form, $y = mx + b$ by solving for y in $x + 2y = 4$.

$$\begin{aligned} x + 2y &= 4 && \text{subtract } x \\ 2y &= -x + 4 && \text{divide by } 2 \\ y &= \frac{-x+4}{2} = \frac{-x}{2} + \frac{4}{2} = -\frac{1}{2}x + 2 \end{aligned}$$

The slope-intercept form of the equation is $y = -\frac{1}{2}x + 2$.

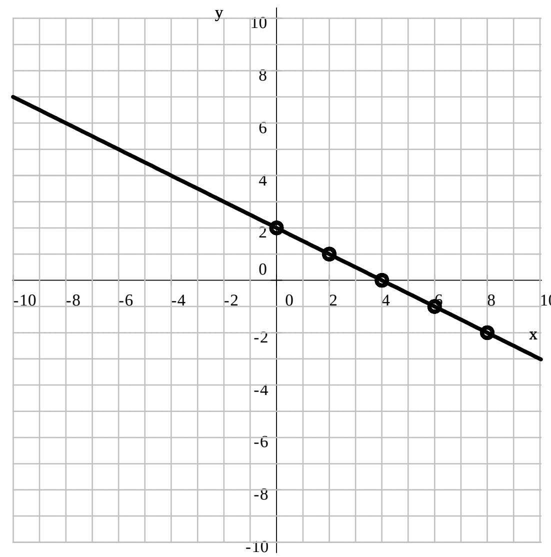
Step 2. We graph first the $y-$intercept of the line, which we obtain by substituting $x = 0$ into the slope-intercept form of the line, $y = -\frac{1}{2}x + 2$. Clearly, the $y-$intercept is $(0, 2)$.

Step 3. Graph additional points using the slope of the line using the slope.

The slope of this line is $m = -\frac{1}{2}$. The denominator tells us how many units we step to the right, and the numerator tells us how many units we step down (since the slope is negative).

$$-\frac{1}{2} = \frac{-1}{2} \implies \frac{1 \text{ down}}{2 \text{ to the right}}$$

We start at the $y-$intercept, $(0, 2)$ and to plot additional points, we step 2 to the right, 1 down. We repeat this process several times to plot enough points.



Example 6. Graph the line $3x - 4y = 12$.

Solution: Step 1. We bring the equation to its slope-intercept form, $y = mx + b$ by solving for y in $3x - 4y = 12$.

$$\begin{aligned} 3x - 4y &= 12 && \text{add } 4y \\ 3x &= 4y + 12 && \text{subtract } 12 \\ 3x - 12 &= 4y && \text{divide by } 4 \\ \frac{3x-12}{4} &= y \\ y &= \frac{3x-12}{4} = \frac{3x}{4} - \frac{12}{4} = \frac{3}{4}x - 3 \end{aligned}$$

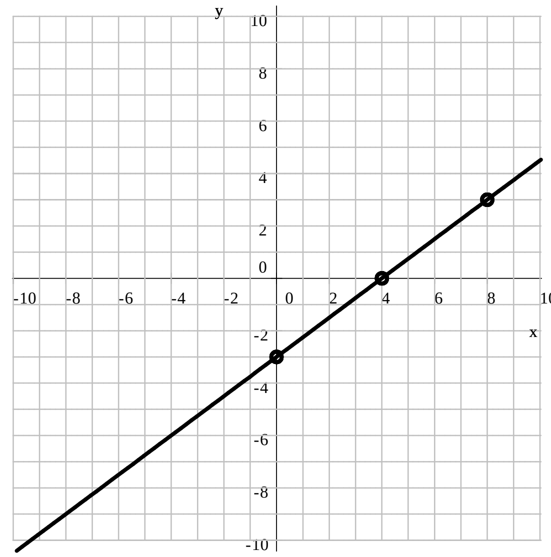
The slope-intercept form of the equation is $y = \frac{3}{4}x - 3$.

Step 2. We graph the $y-$intercept of the line. We obtain the $y-$intercept by substituting $x = 0$ into the equation. Clearly, the $y-$intercept is $(0, -3)$.

Step 3. Graph additional points using the slope of the line.

$$m = \frac{3}{4} \implies \frac{3 \text{ up}}{4 \text{ to the right}}$$

We start at the $y-$intercept, $(0, -3)$ and to plot additional points, we step 4 to the right, and 3 up. We repeat this process several times to plot enough points.



> **Theorem**: Two lines are parallel if and only if they are both vertical or they have the same slope.

In other words, two lines, if they have slopes and are parallel, then the slopes must be the same.

> **Theorem**: Two lines are perpendicular if and only if one is vertical and the other is horizontal, or they both have slopes and they are negative reciprocals, $m_1 m_2 = -1$.

Practice Problems

1. In each case, determine the slope of the line connecting the points given.

   a) $(-2, 7)$ and $(3, -3)$ b) $(7, -5)$ and $(3, -8)$ c) $(3, -2)$ and $(3, 6)$ d) $(2, 7)$ and $(10, 7)$

2. Graph each of the following lines.

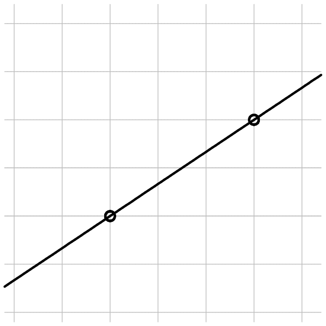
   a) $3x + 2y = 6$ d) $2x - 3y = 10$ g) $3x + 5y = -30$

   b) $x = -4$ e) $y = 1$ h) $2x - y = 7$

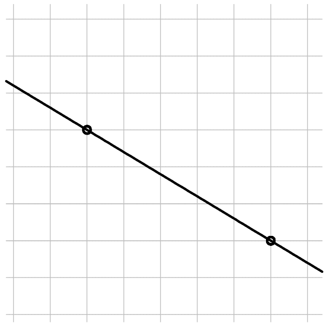
   c) $y = \dfrac{2}{5}x - 3$ f) $y = 3x + 6$ i) $y = \dfrac{1}{3}x$

3. Determine the slope of each of the following lines, based on their graphs.

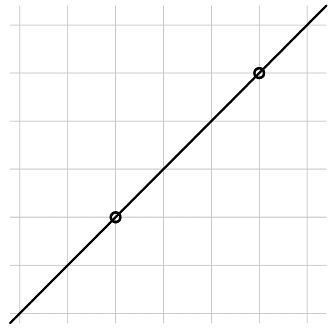
a)

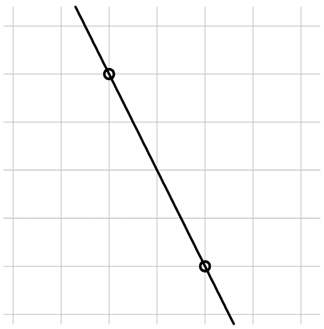


b)

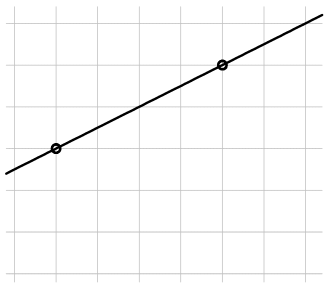


c)

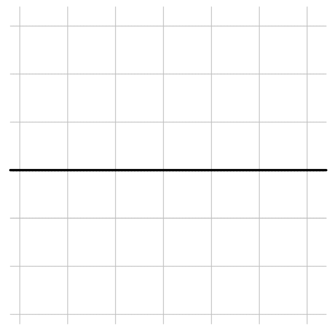




d)



e)

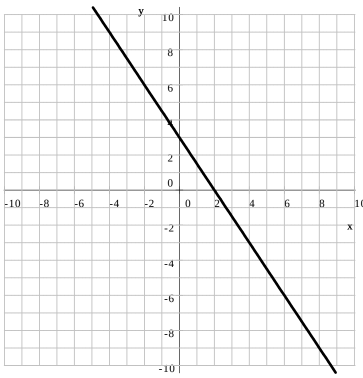


f)

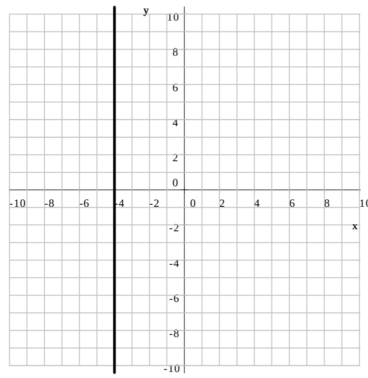
Answers for Practice Problems

1. a) -2 b) $\frac{3}{4}$ c) undefined d) 0

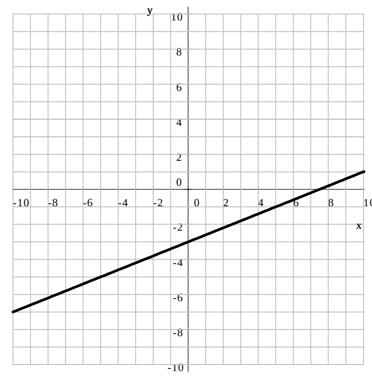
2. a) $3x + 2y = 6$



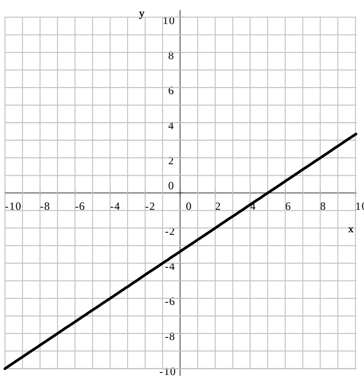
b) $x = -4$



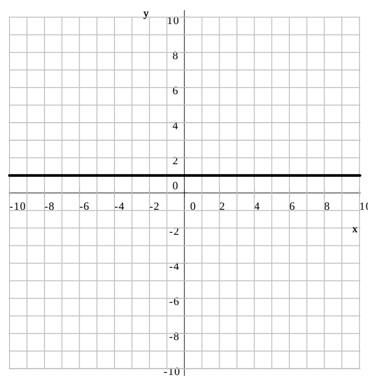
c) $y = \frac{2}{5}x - 3$



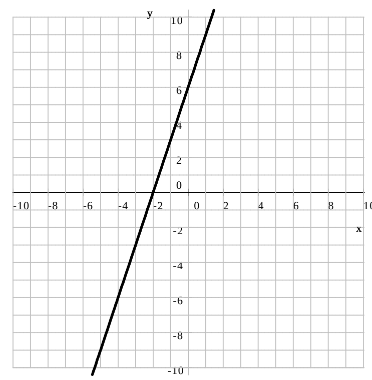
d) $2x - 3y = 10$



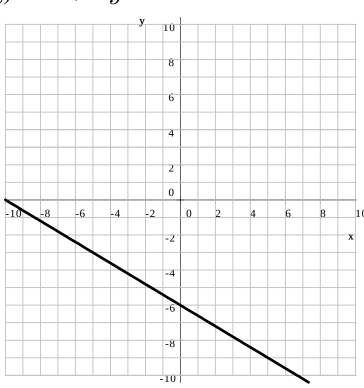
e) $y = 1$



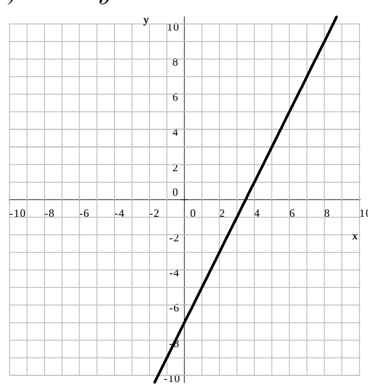
f) $y = 3x + 6$



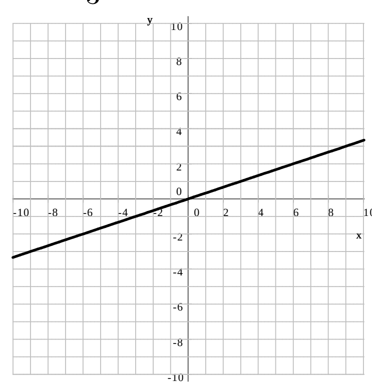
g) $3x + 5y = -30$



h) $2x - y = 7$



i) $y = \frac{1}{3}x$



3. a) $m = \frac{2}{3}$ b) $m = -\frac{3}{5}$ c) $m = 1$ d) $m = -2$ e) $m = \frac{1}{2}$ f) $m = 0$

