The good news is that order of operations on the set of all integers ($\mathbb{Z}$) is the same as on the set of all natural numbers ($\mathbb{N}$). This is not by chance: when mathematicians enlarge our number system, they make certain that the new numbers and operations defined are in harmony with properties of the smaller system. We will refer to this consistent effort as **the expansion principle**. The bad news is that notation became more complex with the introduction of zero and the negative numbers.

The Negative Sign

The negative sign has a dual role: it denotes subtraction or it describes the number after it. This ambiguity never leads to confusion, otherwise it wouldn't be allowed. It is extremely important that we correctly interpret these roles before starting a computation. Here is a neat trick: when facing a negative sign, we should ask ourselves if it could denote subtraction. If it can denote subtraction, then it *does* denote subtraction. If it cannot, then it is a sign to be read as '*negative*' or '*the opposite of*'. A negative sign denoting subtraction is one that is loose between two numbers.

-3	$-(-3)$	$8-3$	$8(-3)$	$8-(-3)$	$8(-(-3))$
negative 3	the opposite of negative 3	subtraction	negative 3	subtract negative 3	8 times the opposite of negative 3

The following situation is often difficult for students.

Example 1. Simplify the expression $20-3(-5)$.

Solution: The first negative sign indicates subtraction, the second one describes 5 as being negative. We start with the multiplication. However, we do not include the subtraction sign in the multiplication. $3(-5)=-15$, and that is what we will subtract.

$$\begin{aligned} 20-3(-5) &= 20-(-15) && \text{to subtract is to add the oposite} \\ &= 20+15 && \text{addition} \\ &= \boxed{35} \end{aligned}$$

Example 2. Consider the expression $20-3(-4-(-2)5)$.

a) Pair the opening and closing parentheses. Classify them as grouping symbols or clarifying prentheses.

b) How many operations are indicated?

c) Perform the indicated operations. For each step, write a separate line.

Solution: a) There are two pairs of parentheses, nested inside each other. The outside pair is a grouping symbol, the one enclosing -2 is a clarifying parentheses.

b) We scan the expression left to right and count the operations. At this point we must be clear on the role of each negative sign. The negative signs before 4 and 2 do not denote subtraction.

1. subtraction between 20 and 3 (although it is not going to be 3 what we are subtracting)
2. multiplication between 3 and the expression in parentheses
3. subtraction between -4 and (-2) (although it is not going to be -2 what we are subtracting)
4. multiplication between -2 and 5

So there are four operations.

c) We will write a separate line for each step. There are two advantages of this. First, at every point we just need to find the *one* operation to execute. Second, each line we write is a new and easier problem we need to solve. Within the parentheses, there is a subtraction and a multiplication. We start with the multiplication $-2 \cdot 5$.

$$\begin{aligned} 20 - 3\left(-4 - (-2)\,5\right) &= 20 - 3\left(-4 - (-10)\right) && \text{to subtract is to add the opposite} \\ &= 20 - 3\left(-4 + 10\right) && \text{addition (also, drop parentheses)} \\ &= 20 - 3 \cdot 6 && \text{multiplication} \\ &= 20 - 18 && \text{subtraction} \\ &= \boxed{2} \end{aligned}$$

The Trouble with Exponents - It's an Order of Operations Thing

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which can also be interpreted as mutliplication by -1. We can interpret -3 as $-1 \cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1 \cdot 3^2$ and $(-1 \cdot 3)^2$. The rest is really just an order of operations problem.
Recall that in our order of operations agreement, exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication.

If there is no parentheses, we have	If we have parentheses, then
$$\begin{aligned} -3^2 &= -1 \cdot 3^2 && \text{exponentiation first} \\ &= -1 \cdot 9 && \text{multiplication} \\ &= -9 \end{aligned}$$	$$\begin{aligned} (-3)^2 &= (-1 \cdot 3)^2 && \text{multiplication first} \\ &= (-3)^2 && \text{square the number } -3 \\ &= 9 \end{aligned}$$

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

-3^2 is the opposite of the square of 3 $\qquad$ $(-3)^2$ is the square of the opposite of 3

Example 3. Simplify each of the given expressions.

a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-\left(-2^2\right)$

Solution: a) $-2^4 = -1 \cdot 2^4 = -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = -1 \cdot 16 = -16$
-2^4 can be read as the opposite of 2^4.

b) $(-2)^4 = (-1 \cdot 2)^4 = (-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2) = (-2)(-2)(-2)(-2) = 16$
$(-2)^4$ can be read as the fourth power of -2.

c) $-1^3 = -1 \cdot 1^3 = -1 \cdot 1 \cdot 1 \cdot 1 = -1$

d) $(-1)^3 = (-1)(-1)(-1) = -1$

e) $-(-2)^2 = -1 \cdot \left((-1 \cdot 2)^2\right) = -1 \cdot \left((-2)^2\right) = -1 \cdot 4 = -4$

f) Careful! The exponent is inside the parentheses. This is squaring 2 and then taking the opposite of the result twice.

$-\left(-2^2\right) = -1 \cdot \left(-1 \cdot 2^2\right) = -1 \cdot (-1 \cdot 4) = -1\,(-4) = 4$

Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Example 4. Evaluate each of the given expressions.

a) $5(-2)^2$ b) $5\left(-2^2\right)$

Solution: We need to pay attention to the subtle difference between the two expressions: the exponent is inside the parentheses in part b. In that second example, the parentheses is not necessary for the exponentiation, without it we would be looking at subtraction.

a) $5(-2)^2 = 5 \cdot 4 = 20$ b) $5\left(-2^2\right) = 5(-4) = -20$

Example 5. Simplify the given expression. Write a separate line for each step. $-3^2 - 2\left(3\left(10 - (-2)^2\right) - 4^2\right)^2$

Solution: We start with the exponentiation in the innermost parentheses.

$$-3^2 - 2\left(3\left(10 - (-2)^2\right) - 4^2\right)^2 =$$

$$\begin{aligned}
&= -3^2 - 2\left(3(10-4) - 4^2\right)^2 && \text{subtraction in innermost parentheses} \\
&= -3^2 - 2\left(3 \cdot 6 - 4^2\right)^2 && \text{exponentiation in parentheses} \\
&= -3^2 - 2(3 \cdot 6 - 16)^2 && \text{multiplication in parentheses} \\
&= -3^2 - 2(18 - 16)^2 && \text{subtraction in parentheses} \\
&= -3^2 - 2 \cdot 2^2 && \text{first exponentiation left to right; } \quad -3^2 = -9 \\
&= -9 - 2 \cdot 2^2 && \text{exponentiation} \\
&= -9 - 2 \cdot 4 && \text{multiplication} \\
&= -9 - 8 && \text{subtraction} \\
&= \boxed{-17}
\end{aligned}$$

The Trouble with the Absolute Value Sign

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems illustrate that very similarly looking problems can actually be quite different. The key to solving these types of is to first pair off the absolute value signs.

Example 6. Evaluate each of the following expressions.

a) $|-5-3| - |7+2|$ b) $||-5-3| - 7| + 2$ c) $-5 - ||3-7| + 2|$ d) $-5|-3 - |7+2||$

Solution: a) Consider the expression $|-5-3| - |7+2|$.

We will not be able to get far in solving this problem without pairing the absolute values signs. Naturally, the first vertical bar can only be an opening sign. The second one can not be an another opening sign, becuase $|-|$ does not make any sense. Therefore, we have two pairs, one after the other. We should indicate the pairing using different colors or different sizes for the two pairs of absolute values signs. Keep in mind, they also serve as grouping symbols. After we paired up the signs, the problem becomes quite easy.

a) $|-5-3|-|7+2| =$

$$
\begin{aligned}
&= \left| -5-3 \right| - |7+2| && \text{subtraction in first parentheses} \\
&= \left| -8 \right| - |7+2| && \text{evaluate the absolute value of } -8 \\
&= 8 - |7+2| && \text{addition in parentheses} \\
&= 8 - |9| && \text{evaluate the absolute value of } 9 \\
&= 8 - 9 = \boxed{-1} && \text{subtraction}
\end{aligned}
$$

b) Consider the expression $||-5-3|-7|+2$.

It is easy to see that the first and second absolute value signs can not be a pair, because || does not make sense. Therefore, since they are at the beginning, they are both opening. This means that we have one pair of absolute value signs nested inside the other. The one that opens last has to close first. After we have established that, the problem becomes much easier.

$$
\begin{aligned}
||-5-3|-7|+2 &= \left| |-5-3|-7 \right| + 2 && \text{subtraction in innermost parentheses} \\
&= \left| |-8|-7 \right| + 2 && \text{evaluate the absolute value of } -8 \\
&= |8-7|+2 && \text{subtraction in parentheses} \\
&= |1|+2 && \text{evaluate the absolute value of } 1 \\
&= 1+2 = \boxed{3} && \text{addition}
\end{aligned}
$$

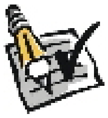
c) Consider the expression $-5-||3-7|+2|$.

We again see that || in front of 3 can not be a pair. Thus they are both opening, and so we have one pair inside the other.

$$
\begin{aligned}
-5-||3-7|+2| &= -5 - \left| |3-7|+2 \right| && \text{subtraction in innermost parentheses} \\
&= -5 - \left| |-4|+2 \right| && \text{evaluate the absolute value of } -4 \\
&= -5 - |4+2| && \text{addition in parentheses} \\
&= -5 - |6| && \text{evaluate the absolute value of } 6 \\
&= -5 - 6 = \boxed{-11} && \text{subtraction}
\end{aligned}
$$

d) Consider the expression $-5\,|-3-|7+2||$. We again see that || after 2 can not be a pair. Thus they are both closing, and so we have one pair inside the other.

$$
\begin{aligned}
-5\,|-3-|7+2|| &= -5 \left| -3-|7+2| \right| && \text{addition in innermost parentheses} \\
&= -5 \left| -3-|9| \right| && \text{evaluate the absolute value of } 9 \\
&= -5\,|-3-9| && \text{subtraction in parentheses} \\
&= -5\,|-12| && \text{evaluate the absolute value of } -12 \\
&= -5 \cdot 12 = \boxed{-60} && \text{multiplication}
\end{aligned}
$$



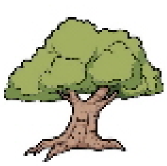
Practice Problems

Simplify each of the follwing expressions by applying the order of operations agreement. Write a separate line for each step.

1. $-2-3\left(1-4^2\right)$
2. $-2-3(1-4)^2$
3. $(-2-3(1-4))^2$
4. $(-2-3)(1-4)^2$
5. $-2(-3(1-4))^2$
6. $\dfrac{-7-(-2)^3-(-1)^4}{-2(5-(-5))}$
7. $\dfrac{-24\div 2(-3)}{5-(-1)^2}$
8. $(2-5)\left(12-(-4)^2\right)$
9. $2-5\left(12-(-4)^2\right)$
10. $13-5(7-10)$
11. $13-5\left|7-10\right|$
12. $\dfrac{12-3(8-5)}{2^4-(-4)^2}$
13. $-3^2-\left((-2)^2-3(-1)^3\right)$
14. $(-3)^2-2\left(3-4\left(5-2^3\right)\right)$
15. $\left(\left(\left(5-(-2)^2\right)^2-3\right)^2+1\right)^2-1$
16. $(-2)^2-(-2)^3-(-2)^4$
17. $-(-5)^2-2\left(3-5^2\right)$
18. $-(-5)^2-2(3-5)^2$
19. $-\left((-5)^2-2(3-5)^2\right)$
20. $\dfrac{5(-2)-3\left|-5+(-1)^2\right|}{-2(3-(-2))-(-1)^2}$
21. $\dfrac{10-2\left(-3-(-1)^3\right)}{(-2)^3+(-1)^4}$
22. $\dfrac{12-2(3(4(5-7)+5)-3)}{-3+2\left(7-(-3)^2\right)-(-1)^3}$
23. $4-\left(6-\left(8-\left(10-\left(12-4^2\right)\right)\right)\right)$

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems can be solved by first pairing off the absolute value signs.

24. $5-2\,||-4+7|-10|$
25. $5\,|-2\,|-4+7||-10$
26. $5-2\,|-4+|7-10||$
27. $5\,|-2-4+|7-10||$
28. $5-|2-|4+7|-10|$
29. $|5-|2-4||+7-10$
30. $|5-2|-|4+7-10|$
31. $||5-2|-4|+7-10$
32. $||5-2|-4+7|-10$



Enrichment

1. We would hope that our order of operations agreement is void of ambiguities. This is not true. In case of the absolute value sign, we cannot tell whether it is opening or closing. Because of this, it is possible to find expressions that can be interpreted correctly in several ways, and the results are different! Evaluate each of the given expressions correctly in two different ways. What are your results?

   a) $|5-2|-4+7\,|-10|$ b) $2\,|-1-5|-3\,|-4|$

2. Enter pairs of parentheses or absolute value signs into the expression on the left-hand side to make the follwing statements true.

   a) $5-2-3\;{}^2-4-7\;{}^2+2=9$ b) $5-2-3\;{}^2-4-7\;{}^2+2=-9$

Answers

Practice Problems

1. 43 2. -29 3. 49 4. -45 5. -162 6. 0 7. 9 8. 12 9. 22 10. 28 11. -2

12. undefined 13. -16 14. -21 15. 24 16. -4 17. 19 18. -33 19. -17 20. 2

21. -2 22. -6 23. -8 24. -9 25. 20 26. 3 27. 15 28. -14 29. 0 30. 2

31. -2 32. -4

Enrichment

1. a) 69 and 11   b) 0 and 40

   The pairs are colored differently. If your printout is black and white, look at the document at your computer.

   a) $|5-2|-4+7\,|-10|=69$ and $|5-2\,|-4+7|-10|=11$

   b) $2\,|-1-5|-3\,|-4|=0$ and $2\,|-1-5\,|-3|-4|=40$

2. a) $5-2\left((-3)^2-\left((4-7)^2+2\right)\right)=9$ b) $5-\left((2-3)^2-(4-7)\right)^2+2=-9$

