This handout will provide a quick introduction into algebraic expressions.

Definition: A **numerical expression** is an expression that combines numbers and operations.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\dfrac{12}{3+1}$ and $5^2 - 2^2$ and $-|-5|$. We can **evaluate** numerical expressions by correctly applying the order of operations agreement. It is important that we clearly understand notation.

Example 1: Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\dfrac{12}{3+1}$ c) $3^2 + 2^2$ d) $(3+2)^2$ e) -3^2 f) $(-3)^2$ g) $-|-5|$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?) $-\dfrac{12}{3+1} = -\dfrac{12}{4} = \boxed{-6}$

c) $3^2 + 2^2 = 9 + 4 = \boxed{13}$

d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing $3^2 + 2^2$ with $(3+2)^2$ is called the "Freshman's Dream Error".

e) $-3^2 = \boxed{-9}$

f) $(-3)^2 = \boxed{9}$

Note: In looking at -3^2 and $(-3)^2$, we can interpret the minus sign in front of 3 as 'the opposite of'. That is the same as multiplication by -1. Now we can apply order of opertaions, and exponentiation comes before multiplication.

$$-3^2 = -1 \cdot 3^2 = -1 \cdot 9 = -9 \quad \text{but} \quad (-3)^2 = (-3)(-3) = 9$$

In the case of -3^2, we take the opposite of the square of three.
In the case of $(-3)^2$, we square the opposite of three.

g) $-|-5| = \boxed{-5}$ This is a perfect example that two minuses don't always make a plus. What happens here?

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables always represent numbers, so they are subjects to the same rules as numbers. For example, $3x^2 - 1$ is an algebraic expression. So are $-x + 3$ and $2a - b$ and $5y + 3$. We can not automatically evaluate an algebraic expression because we often do not know the value of the variables. For example, the expression $3x^2 - 1$ has different values for different values of x.

Example 2: Evaluate the algebraic expression $3x^2 - 1$ given the values of x.

a) $x = 2$ b) $x = 0$ c) $x = -1$ d) $x = 5$

Solution: a) If $x = 2$, then $3x^2 - 1 = 3 \cdot 2^2 - 1 = 3 \cdot 4 - 1 = 12 - 1 = \boxed{11}$

b) If $x = 0$, then $3x^2 - 1 = 3 \cdot 0^2 - 1 = 3 \cdot 0 - 1 = 0 - 1 = \boxed{-1}$

c) If $x = -1$, then $3x^2 - 1 = 3(-1)^2 - 1 = 3 \cdot 1 - 1 = 3 - 1 = \boxed{2}$

d) If $x = 5$, then $3x^2 - 1 = 3(5)^2 - 1 = 3 \cdot 25 - 1 = 75 - 1 = \boxed{74}$

When we don't know the value of a variable, we often need to **simplify** algebraic expressions. Consider, for example, the algebraic expression $2x + 3x$. We can simplify $2x + 3x$ and just write $5x$ instead. This is also called **combining like terms.**

Example 3: Simplify each of the following by combining like terms.
a) $3x-2-10x+x+7$ b) $3m-2n-4-8m+n+1$ c) $ab-a+b$ d) $p+q+p-q$

Solution: a) $3x$, $-10x$, and x are like terms, and -2 and 7 are like terms.
To combine like terms, we add the numbers (sign included!) that are multiplying the variable(s). Such a number is called the **coefficient**. When combining like terms, we add the coefficients.

To combine $3x$, $-10x$, and x, we add the coefficients: $3x-10x+x=(3-10+1)\,x=-6x$
Caution! It is a common mistake to misinterpret $3-10+1$ as $3-11$. Not so!

To combine -2 and 7, we just add and so we get 5. The entire process can be done mentally, so our computation will look like this:

$$3x-2-10x+x+7=\boxed{-6x+5}$$

b) $3m-2n-4-8m+n+1=3m-8m-2n+n-4+1=\boxed{-5m-n-3}$
-3, $-n$, and $-5m$ are unlike terms and so this expression can not be further simplified.

c) $\boxed{ab-a+b}$ since all three terms are unlike, so this expression can not be simplified.

d) $p+q+p-q=p+p+q-q=\boxed{2p}$

In the last example, as we simplified the expression, one of the variables disappeared. This special and often celebrated case of combining like terms is what we call **cancellation**.

> To add two or more algebraic expressions, we drop parentheses and combine like terms.

Example 4: Add the algebraic expressions as indicated.

a) $(3a-5b)+(2a-b)$ b) $(-m+3n-4)+(5m-n-4)$ c) $(3y+5)+(3y-5)$

Solution: a) $(3a-5b)+(2a-b)=3a-5b+2a-b=3a+2a-5b-b=\boxed{5a-6b}$

b) $(-m+3n-4)+(5m-n-4)=-m+5m+3n-n-4-4=\boxed{4m+2n-8}$

c) $(3y+5)+(3y-5)=3y+5+3y-5=3y+3y+5-5=\boxed{6y}$

> To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law:
>
> $a\,(b+c)=ab+ac$ for all numbers a, b, and c.

Example 5: Expand the products as indicated.

a) $3\,(5a-b+1)$ b) $-1\left(-x^2+3x-4\right)$ c) $5x\,(2a-x)$ d) $-ab\,(3a-5b-1)$ e) $-4\,(3m-5)$

Solution: a) $3\,(5a-b+1)=\boxed{15a-3b+3}$

b) $-1\,(-x+3y-4)=\boxed{x-3y+4}$

Note that multipliction by -1 means we change the sign in front of each term.

c) $5x\,(2a-3)=5x\cdot 2a-5x\cdot 3=10ax-15x=\boxed{10ax-15x}$

d) $-ab\,(3a-5b-1)=-ab\cdot 3a-ab\,(-5b)-ab\,(-1)=\boxed{-3a^2b+5ab^2+ab}$

e) $-4\,(3m-5)=\boxed{-12m+20}$

To subtract an algebraic expression, we add the opposite.

Example 6: Perform the subtractions between algebraic expressions as indicated.

a) $(3a-5b)-(2a-b)$ b) $(3y+5)-(3y-5)$

Solution: a) We apply one fundamental fact of algebra: *To subtract is to add the opposite.* The opposite is always obtained by multiplication by -1. And this multiplication by -1 means we need to distribute -1. We subtract the entire expression, not just its first term. Here is the argument, broken down to logical steps.

$$\begin{aligned}
(3a-5b)-(2a-b) &= \quad \text{to subtract is to add the opposite,}\\
(3a-5b)+(-1)(2a-b) &= \quad \text{the opposite is obtained by multiplying by } -1 \text{ (careful with the distributive law)}\\
(3a-5b)+(-2a+b) &= \quad \text{we add the algebraic expressions by dropping the parentheses}\\
3a-5b-2a+b &= \quad \text{and combine like terms}\\
3a-2a-5b+b &= \boxed{a-4b}
\end{aligned}$$

But this is way too much writing. While the idea is the same, our computation usually looks like this:

$$(3a-5b)-(2a-b)=3a-5b-2a+b=a-4b$$

Careful! When computing on paper, we advise *not to subtract mentally*. To subtract is to add the opposite. Take the time of writing down the opposite, and then *add* mentally.

b) $(3y+5)-(3y-5)=(3y+5)+(-3y+5)=\boxed{10}$

We can now combine more complicated expressions.

Example 7: Simplify each of the following expressions.

a) $4(a-2b+1)-5(2a-b-1)$ b) $6(2y+1)-5(3y-5)$ c) $3(5x-2)-5(3x+1)$

Solution: a) We apply the distributive law and then combine like terms.

$$4(a-2b+1)-5(2a-b-1)=4a-8b+4-10a+5b+5=\boxed{-6a-3b+9}$$

b) $6(2y+1)-5(3y-5)=12y+6-15y+25=\boxed{-3y+31}$

c) $3(5x-2)-5(3x+1)=15x-6-15x-5=\boxed{-11}$

The last example illustrates the benefits of algebra. If we were asked to evaluate the expression $3(5x-2)-5(3x+1)$ when $x=8,-20$, or -97, or any other number, we might be computing for minutes, but the result will always be -11. In other words, this expression is **equivalent** to -11. It is a natural instinct to present and think of expressions in their simplest possible form.

Discussion: Explain how $2x+3x=5x$ can be explained in terms of the distributive law.

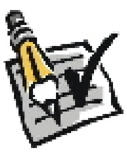
Note: The equation $3a+2a=5a$ is an identity. An identity is an equation for which all numbers that can be substituted into both sides are solution.

$$\begin{aligned}
\text{If } a &= 5, \text{ then } 3\cdot5+2\cdot5=15+10=25=5\cdot5\\
\text{If } a &= -2, \text{ then } 3(-2)+2(-2)=-6+(-4)=-10=5(-2)
\end{aligned}$$

When we simplify algebraic expressions, equations such as $5x-2x=3x$ are always identities.

Sample Problems

1. Evaluate each of the following numerical expressions.
   a) $2-5(3-7)$ b) $24-10+2$ c) -4^2 d) $(-4)^2$ e) $|3|-|8|$ f) $|3-8|$
2. Evaluate each of the following algebrac expressions with the value(s) given.
   a) $-x^2-5x+2$ if $x=-2$ b) $-16t^2+32t+240$ if $t=3$
3. Add the algebraic expressions as indicated.
   a) $(3x-5y)+(2x+4y)$ b) $(2a-5b+3)+(-a-8b+3)$
4. Multiply the algebraic expressions by a number as indicated.
   a) $3(2x+4y-5)$ b) $-5(2a-b+8)$ c) $-1(-p+3q-8m+6)$ d) $-(-a+3b-7)$
5. Subtract the algebraic expressions as indicated.
   a) $(2a+b)-(a-b)$ c) $(2m-5n+3)-(-m-8n+3)$ e) $(3a-2)-(1-4a)$
   b) $(3x-5y)-(2x+4y)$ d) $(2a-2b)-(b-a)$
6. Simplify each of the following.
   a) $3(x-5)-5(x-1)$ b) $4(2a-b)-3(5a-2b)$
   c) $2(a-2b)+3(5b-2a)-4(2b-a)$ d) $-(3a-2)-(1-4a)$



Practice Problems

1. Evaluate each of the following numerical expressions.
   a) $24-5+1$ b) $24\div 3\cdot 2$ c) -1^2 d) $(-1)^2$ e) $-|4|-|7|$ f) $-|4-7|$ g) 6^2-4^2 h) $(6-4)^2$
2. Evaluate each of the following algebrac expressions with the value(s) given.
   a) $3x^2-x+7$ if $x=-1$ b) $-a+5b$ if $a=3$ and $b=-2$ c) $\dfrac{x^x-1}{x-1}$ if $x=2$
3. Add the algebraic expressions as indicated.
   a) $(2a-7y+1)+(2a-7y-1)$ c) $(4p-5q+1)+(5p-4q-2)$ e) $(-5y+8)+(5y-8)$
   b) $(-2a-5)+(3a+5)$ d) $(2x-3y+8)+(-2x-3y-8)$
4. Multiply the algebraic expressions by a number as indicated.
   a) $5(-3a+b-5)$ b) $0(2x-7y+8z)$ c) $-6(n-3m-8)$ d) $-(p-5q+3r-1)$
5. Subtract the algebraic expressions as indicated.
   a) $(2a-7y)-(2a-7y)$ c) $(2x-5y-1)-(-y+1)$
   b) $(-2a-5b-1)-(3a+5b-1)$ d) $(2m-3n+8)-(-2m-3n-8)$
6. Simplify each of the following.
   a) $(x+1)-(x-1)$ c) $-2+x-3(x-1)-(1-2x)$
   b) $2(x-1)-3(x-7)$ d) $-2(a-3b+c)-3(a-b+4c)$

Answers

Sample Problems

1. a) 22 b) 16 c) -16 d) 16 e) -5 f) 5 2. a) 8 b) 192 3. a) $5x-y$ b) $a-13b+6$
4. a) $6x+12y-15$ b) $-10a+5b-40$ c) $p-3q+8m-6$ d) $a-3b+7$
5. a) $a+2b$ b) $x-9y$ c) $3m+3n$ d) $3a-3b$ e) $7a-3$
6. a) $-2x-10$ b) $-7a+2b$ c) $3b$ d) $a+1$

Practice Problems

1. a) 20 b) 16 c) -1 d) 1 e) -11 f) -3 g) 20 h) 4 i) 13 j) 17 2. a) 11 b) -13 c) 3
3. a) $4a-14y$ b) a c) $9p-9q-1$ d) $-6y$ e) 0
4. a) $5b-15a-25$ b) 0 c) $-6n+18m+48$ d) $-p+5q-3r+1$
5. a) 0 b) $-5a-10b$ c) $2x-4y-2$ d) $4m+16$
6. a) 2 b) $-x+19$ c) 0 d) $-5a+9b-14c$

Solutions - Sample Problems

1. Evaluate each of the following numerical expressions.

a) $2-5(3-7)$

Solution: We will apply order of operations. First we perform the subtraction in the parentheses.

$$\begin{aligned} 2-5(3-7) &= && \text{subtraction in parentheses} \\ 2-5(-4) &= && \text{multiplication} \\ 2-(-20) &= && \text{subtraction} \\ 2+20 &= \boxed{22} \end{aligned}$$

b) $24-10+2$

Solution: It is NOT true that addition comes before subtraction. Addition and subtraction are equally strong, so between those two, we perform them left to right. First come, first served.

$$24-10+2 = 14+2 = \boxed{16}$$

c) -4^2

Solution: as it was discussed before, -4^2 is quite different from $(-4)^2$. This is $-1 \cdot 4^2 = \boxed{-16}$.

d) $(-4)^2$

This is when -4 is squared. So $(-4)^2 = -4(-4) = \boxed{16}$

e) $|3|-|8|$

Solution: We subtract the absolute value of 8 from the absolute value of 3. So $|3|-|8| = 3-8 = \boxed{-5}$

f) $|3-8|$

Solution: This is the absolute value of the difference. Absolute value signs also function of grouping symbols (i.e. parentheses) to overwrite the usual order of operations. So $|3-8| = |-5| = \boxed{5}$

2. Evaluate each of the following algebrac expressions with the value(s) given.

a) $-x^2 - 5x + 2$ if $x = -2$

Solution: We substitute -2 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$-x^2 - 5x + 2 = -(-2)^2 - 5(-2) + 2$$

According to order of operations, we perform the exponentiation first.

$$-(-2)^2 - 5(-2) + 2 = -4 - 5(-2) + 2 = -4 - (-10) + 2 = -4 + 10 + 2 = 6 + 2 = \boxed{8}$$

Why don't the two minuses make a plus in $-(-2)^2$? They do, it's just that there are three minus signs and not two: $-(-2)^2 = -1(-2)(-2)$.

b) $-16t^2 + 32t + 240$ if $t = 3$

Solution: We substitute 3 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$\begin{aligned} -16t^2 + 32t + 240 &= -16(3)^2 + 32(3) + 240 \\ &= -16 \cdot 9 + 32 \cdot 3 + 240 \\ &= -144 + 96 + 240 = -48 + 240 = \boxed{192} \end{aligned}$$

3. Add the algebraic expressions as indicated.

a) $(3x - 5y) + (2x + 4y)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (3x - 5y) + (2x + 4y) &= \\ 3x - 5y + 2x + 4y &= \qquad \text{organize like terms together} \\ 3x + 2x - 5y + 4y &= \qquad 3 + 2 = 5 \quad \text{and} \quad -5 + 4 = -1 \\ &= \boxed{5x - y} \end{aligned}$$

b) $(2a - 5b + 3) + (-a - 8b + 3)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (2a - 5b + 3) + (-a - 8b + 3) &= \\ 2a - 5b + 3 - a - 8b + 3 &= \\ 2a - a - 5b - 8b + 3 + 3 &= \qquad 2 - 1 = 1, \quad -5 - 8 = -13, \quad \text{and} \quad 3 + 3 = 6 \\ 1a - 13b + 6 &= \boxed{a - 13b + 6} \end{aligned}$$

4. Multiply the algebraic expressions by a number as indicated.

a) $3(2x + 4y - 5)$

Solution: We distribute 3.

$$3(2x + 4y - 5) = \boxed{6x + 12y - 15}$$

b) $-5(2a - b + 8)$

Solution: We distribute -5.

$$-5(2a - b + 8) = \boxed{-10a + 5b - 40}$$

c) $-1(-p + 3q - 8m + 6)$

Solution: We distribute -1.

$$-1(-p + 3q - 8m + 6) = \boxed{p - 3q + 8m - 6}$$

d) $-(-a+3b-7)$

Solution: The notation here indicates multiplication by -1, which is the same as taking the opposite of a quantity. We distribute -1.

$$-1(-a+3b-7) = \boxed{a-3+7}$$

5. Subtract the algebraic expressions as indicated.

a) $(2a+b)-(a-b)$

Solution: To subtract is to add the opposite. The opposite of $a-b$ is $-a+b$ since

$$-1(a-b) = -a+b$$

Thus
$$\begin{aligned} (2a+b)-(a-b) &= (2a+b)+(-a+b) && \text{drop parentheses} \\ &= 2a+b-a+b && \text{combine like terms} \\ &= \boxed{a+2b} \end{aligned}$$

b) $(3x-5y)-(2x+4y)$

Solution: To subtract is to add the opposite. The opposite of $2x+4y$ is $-2x-4y$ since

$$-1(2x+4y) = -2x-4y$$

$$\begin{aligned} (3x-5y)-(2x+4y) &= (3x-5y)+(-2x-4y) && \text{to subtract is to add the opposite} \\ &= 3x-5y-2x-4y && \text{drop parentheses, combine like terms} \\ &= \boxed{x-9y} \end{aligned}$$

c) $(2m-5n+3)-(-m-8n+3)$

Solution: To subtract is to add the opposite. The opposite of $-m-8n+3$ is $m+8n-3$ since

$$-1(-m-8n+3) = m+8n-3$$

$$\begin{aligned} (2m-5n+3)-(-m-8n+3) &= && \text{to subtract is to add the opposite} \\ (2m-5n+3)+(m+8n-3) &= && \text{drop parentheses, combine like terms} \\ 2m-5n+3+m+8n-3 &= \boxed{3m+3n} \end{aligned}$$

d) $(2a-2b)-(b-a)$

Solution: To subtract is to add the opposite. The opposite of $b-a$ is $-b+a$ since

$$-1(b-a) = -b+a$$

$$\begin{aligned} (2a-2b)-(b-a) &= (2a-2b)+(-b+a) && \text{to subtract is to add the opposite} \\ &= 2a-2b-b+a && \text{drop parentheses, combine like terms} \\ &= \boxed{3a-3b} \end{aligned}$$

e) $(3a-2)-(1-4a)$

Solution: To subtract is to add the opposite. The opposite of $1-4a$ is $-1+4a$ since

$$-1(1-4a) = -1+4a$$

$$\begin{aligned} (3a-2)-(1-4a) &= && \text{to subtract is to add the opposite} \\ (3a-2)+(-1+4a) &= && \text{drop parentheses, combine like terms} \\ 3a-2-1+4a &= \boxed{7a-3} \end{aligned}$$

6. Simplify each of the following.

a) $3(x-5)-5(x-1)$

Solution: We apply the law of distributivity and combine like terms. Notice that the last term is $-5(-1)=5$.

$$\begin{aligned} 3(x-5)-5(x-1) &= && \text{apply the distributie law} \\ 3x-15-5x+5 &= && \text{combine like terms} \\ &= \boxed{-2x-10} \end{aligned}$$

b) $4(2a-b)-3(5a-2b)$

Solution: We apply the law of distributivity and combine like terms

$$\begin{aligned} 4(2a-b)-3(5a-2b) &= 8a-4b-15a+6b \\ &= \boxed{-7a+2b} \end{aligned}$$

c) $2(a-2b)+3(5b-2a)-4(2b-a)$

Solution: We apply the law of distributivity and combine like terms

$$2(a-2b)+3(5b-2a)-4(2b-a)=2a-4b+15b-6a-8b+4a=\boxed{3b}$$

d) $-(3a-2)-(1-4a)$

Solution:

$$\begin{aligned} -(3a-2)-(1-4a) &= -1(3a-2)-1(1-4a) && \text{multiplication} \\ &= -3a+2-1+4a && \text{combine like terms} \\ &= \boxed{a+1} \end{aligned}$$

