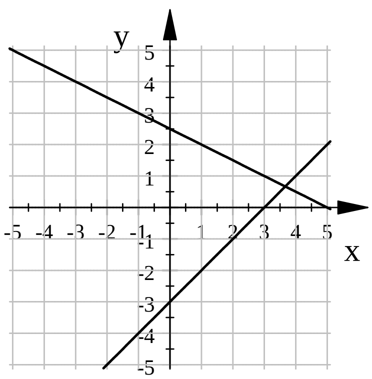
Recall that linear equations in x and y has many, many solutions, and one way to meaningfully represent these solutions is to graph them. Every linear equation can be graphed, and every point on the graph has coordinates that form a solution of the equation.

Therefore, a system of two linear equations can also be graphed, and the intersection point is a point whose coordinates form a simultaneous solution of both equations.

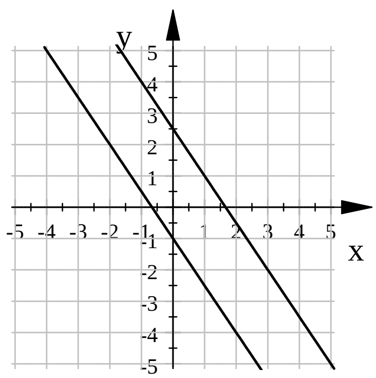
Most systems of linear equations have a unique solution (x, y). However, this is not always the case.

If we think of a system of linear equations as two lines, it is clear that geometrically, there are three distinct ways two lines in a plane can behave.

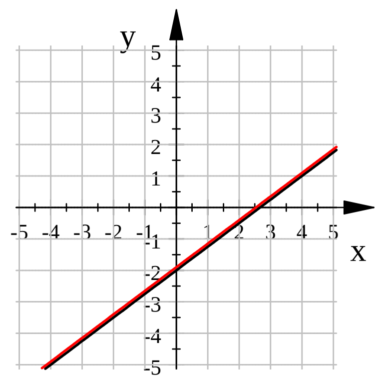
One intersection point



No intersection point



All points on the line are intersection points



Until now, we have only seen linear systems witha unique solution, corresponding to two lines intersecting each other in a uinque point. However, there are linear systems that end up with different results.

Example 1. Solve the given system of linear equations. $\begin{cases} x - 2y = 6 \\ y = \dfrac{1}{2}x + 1 \end{cases}$

Solution: Since the second equation is already solved for y, we will use substitution, but elimination would also work fine. We substitute $y = \dfrac{1}{2}x + 1$ into the first equation and solve the linear equation for x.

$$\begin{aligned} x - 2\left(\frac{1}{2}x + 1\right) &= 6 && \text{distribute 2} \\ x - x - 2 &= 6 && \text{combine like terms} \\ -2 &= 6 \end{aligned}$$

As we solve the equation for x, it disappears from the equation and we are left with an unconditionally false equation. We have seen this before when we solved linear equations. An equation such as $-2 = 6$ is called a contradiction, and it has no solution. What does this result mean for a system?

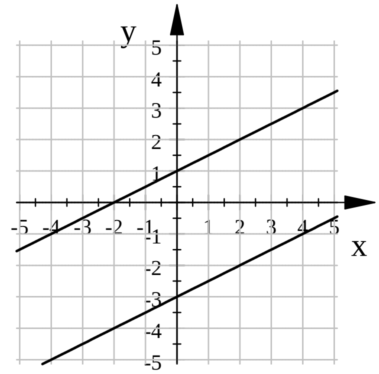
Let us solve the first equation for y.

$$\begin{aligned} x - 2y &= 6 && \text{add } 2y \\ x &= 2y + 6 && \text{subtract 6} \\ x - 6 &= 2y && \text{divide by 2} \\ \frac{x-6}{2} &= y \implies && y = \frac{x}{2} - \frac{6}{2} = \frac{1}{2}x - 3 \end{aligned}$$

Let us look again at the system, but this time from a geometric point of view. Both equations represent a straight line.

$$\begin{cases} y = \dfrac{1}{2}x - 3 \\ y = \dfrac{1}{2}x + 1 \end{cases}$$

These lines are parallel because they have the same slope, $m = \dfrac{1}{2}$, and parallel lines have no intersection points.



In case of such a system, the last line is an unconditionally false equation, a contradiction. Such a system is called an **inconsistent system**, and there is no solution of it.

Example 2. Solve the given system of linear equations. $\begin{cases} 2x + 6y = -12 \\ y = -\dfrac{1}{3}x - 2 \end{cases}$

Solution: Since the second equation is already solved for y, we will use substitution, but elimination would also work fine. We substitute $y = -\dfrac{1}{3}x - 2$ into the first equation and solve the linear equation for x.

$$\begin{aligned} 2x + 6\left(-\frac{1}{3}x - 2\right) &= -12 && \text{distribute } 6 \\ 2x - 2x - 12 &= -12 && \text{combine like terms} \\ -12 &= -12 \end{aligned}$$

As we solve the equation for x, it disappears from the equation again, but this time we are left with an unconditionally true equation. We have seen this before when we solved linear equations. Such an eqaution is called an identity, and all real numbers are solutions of it.

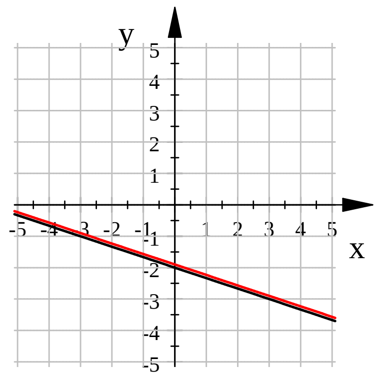
Just as before, we solve the first equation for y.

$$\begin{aligned} 2x + 6y &= -12 && \text{subtract } 2x \\ 6y &= -2x - 12 && \text{divide by } 6 \\ y &= \frac{-2x - 12}{6} && \text{divide by } 2 \\ y &= \frac{-2x - 12}{6} = -\frac{2x}{6} - \frac{12}{6} = -\frac{1}{3}x - 2 \end{aligned}$$

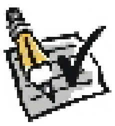
So, the linear system now look like this: $\begin{cases} y = -\dfrac{1}{3}x - 2 \\ y = -\dfrac{1}{3}x - 2 \end{cases}$

$$\begin{cases} y = -\dfrac{1}{3}x - 2 \\ \\ y = -\dfrac{1}{3}x - 2 \end{cases}$$

These lines are identical and so every point on the line is a solution. We can express this as the set of points $\left(x, -\dfrac{1}{3}x - 2\right)$.



In case of such a system, the last line is an unconditionally true equation, an identity. Such a system is called a **dependent system**, and all points on the line are solutions.



Practice Problems

1. Solve each of the following systems of linear equations.

a) $\begin{cases} x + 3y = 11 \\ 12y = -4x + 7 \end{cases}$ c) $\begin{cases} (x-2)^2 + y = (x+1)^2 - y \\ 2y = 6x - 3 \end{cases}$

b) $\begin{cases} 4y = 6x + 10 \\ 3x - 2y = -5 \end{cases}$ d) $\begin{cases} 5x - 6y = -10 \\ \dfrac{1}{2}x - \dfrac{3}{5}y = 1 \end{cases}$

2. Consider the linear system $\begin{cases} 2x + 3y = A \\ x = -\dfrac{3}{2}y - 12 \end{cases}$. Find the value of A so that the system

a) has no solution b) has infinitely many solutions

Answers

Practice Problems

1. Solve each of the following systems of linear equations.

   a) There is no solution. b) All points $\left(x, \dfrac{3}{2}x + \dfrac{5}{2}\right)$ are solution

   c) All points $\left(x, 3x - \dfrac{3}{2}\right)$ are solution d) There is no solution.

2. a) $A = -24$ b) A can be any number except for -24

