Equations that have absolute values in them have interesting properties. Some will remind us to quadratic equations, some will be fundamentally different. Let us start with the most basic forms of such eqautions.

Example 1. Solve each of the given equations.

a) $|x| = 5$ b) $|x| = 0$ c) $|x| = -6$

Solution: a) How can the absolute value of a number be 5? There are only two ways: if the number is 5, or if the number is -5. There are no other ways. Therefore, the solution of $|x| = 5$ is $\boxed{-5, \text{ and } 5}$.

b) How can the absolute value of a number be 0? There is only one way: if the number is 0. Therefore, the solution of $|x| = 0$ is $\boxed{x = 0}$.

c) How can the absolute value of a number be -5? Absolute values are never negative, so it is not possible. Therefore, this equation has $\boxed{\text{no solution}}$.

These basic equations with absolute values in them remind us to quadratic equations in the sense that there might be two, one, or no solution. More complicated equations can be solved by applying these ideas to expressions inside absolute value signs.

Example 2. Solve each of the given equations.

a) $|2x - 7| = 5$ b) $|x - 2| + 4 = 16$

Solution: a) We need to apply the same idea. This time, not on x, but on the expressions inside the absolute value sign. Let us think of the expression $2x - 7$ as a single quantity, say A. How can the absolute value of a number be 5? There are only two ways: if the number is 5, or if the number is -5. But it is not x that needs to be 5 or -5, but rather $A = 2x - 7$. This way, we will trade in one absolute value equation to two linear equations.

$$|2x - 7| = 5 \quad \text{if and only if} \quad 2x - 7 = 5 \quad \text{or} \quad 2x - 7 = -5$$

We solve both linear equations for x.

$2x - 7 = 5$	add 7	$2x - 7 = -5$	add 7
$2x = 12$	divide by 2	$2x = 2$	divide by 2
$x = 6$		$x = 1$	

This equation seems to have two solutions. We check:

if $x = 6$, then LHS $= |2 \cdot 6 - 7| = |12 - 7| = |5| = 5 =$ RHS ✓

And if $x = 1$, then LHS $= |2 \cdot 1 - 7| = |2 - 7| = |-5| = 5 =$ RHS ✓. So the solutions are $\boxed{1 \text{ and } 6}$.

Discussion: Explain why the conclusion shown below is wrong.

$$|2x - 7| = 5 \quad \Longrightarrow \quad 2x + 7 = 5$$

b) In the previous example, the symmetry of 5 and -5 is no longer around x, but around the expression within the absolute value sign. We can exchange the one equation with absolute value to a pair of linear equations **only when the absolute value expression is isolated on one side of the equation**.

$$|x-2|+4=16 \qquad \text{subtract } 4$$

$$|x-2|=12$$

$$|x-2|=12 \quad \text{if and only if} \quad x-2=12 \text{ or } x-2=-12$$

We solve both linear equations for x.

$x-2=12$ add 2	$x-2=-12$ add 2
$x=14$	$x=-10$

We again have two solutions. We check:

if $x=14$, then LHS $=|14-2|+4=|12|+4=12+4=16=$ RHS ✓

And if $x=-10$, then LHS $=|-10-2|+4=|-12|+4=12+4=16=$ RHS ✓

So the solutions are $\boxed{-10 \text{ and } 14}$.

Discussion: Explain why the conclusion shown below is wrong.

$$|x-12|+4=16 \quad \Longrightarrow \quad x-2+4=16 \quad \text{or} \quad x-2+4=-16$$

Example 3. Solve each of the given equations.

a) $3|2x-10|+7=19$ b) $3-\left|\frac{1}{2}x-5\right|=-10$ c) $8+3|5x-4|=2$

Solution: a) We can exchange the one equation with absolute values to a pair of linear equations only when the absolute value expression is isolated on one side ofthe equation. Let us introduce a new variable, say A for the absolute value expression. Let $A=|2x-10|$. We first solve for A.

$3|2x-10|+7=19$ becomes $3A+7=19$

$$\begin{aligned} 3A+7 &= 19 && \text{subtract } 7 \\ 3A &= 12 && \text{divide by } 3 \\ A &= 4 && \text{recall that } A=|2x-10| \\ |2x-10| &= 4 && \Longrightarrow \quad 2x-10=4 \text{ or } 2x-10=-4 \end{aligned}$$

We solve both linear equations for x.

$2x-10=4$ add 10	$2x-10=-4$ add 10
$2x=14$ divide by 2	$2x=6$ divide by 2
$x=7$	$x=3$

This equation seems too has two solutions. We check:

if $x=7$, then LHS $=3\left|2\cdot 7-10\right|+7=3\left|14-10\right|+7=3\left|4\right|+7=3\cdot 4+7=12+7=19=$ RHS ✓

And if $x=3$, then LHS $=3\left|2\cdot 3-10\right|+7=3\left|6-10\right|+7=3\left|-4\right|+7=3\cdot 4+7=12+7=19=$ RHS ✓.

So the solutions are $\boxed{3 \text{ and } 7}$.

b) We first need to eisolate the expression with the absolute value sign. This means that we first solve for $\left|\frac{1}{2}x-5\right|$ in $3-\left|\frac{1}{2}x-5\right|=-10$. We can do that with or without introducing a new variable for $\left|\frac{1}{2}x-5\right|$.

$$\begin{aligned} 3-\left|\frac{1}{2}x-5\right| &= -10 && \text{add } \left|\frac{1}{2}x-5\right| \\ 3 &= -10+\left|\frac{1}{2}x-5\right| && \text{add } 10 \\ 13 &= \left|\frac{1}{2}x-5\right| \end{aligned}$$

Now we can trade the single equation to a pair of linear equations and then solve for z in each.

$$13=\left|\frac{1}{2}x-5\right| \quad\Longrightarrow\quad \frac{1}{2}x-5=13 \quad \text{or} \quad \frac{1}{2}x-5=-13$$

$$\begin{aligned} \frac{1}{2}x-5 &= 13 && \text{add } 5 \\ \frac{1}{2}x &= 18 && \text{multiply by } 2 \\ x &= 36 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}x-5 &= -13 && \text{add } 5 \\ \frac{1}{2}x &= -8 && \text{multiply by } 2 \\ x &= -16 \end{aligned}$$

We check both solutions:

if $x=-16$, then LHS $=3-\left|\frac{1}{2}(-16)-5\right|=3-\left|-8-5\right|=3-\left|-13\right|=3-13=-10=$ RHS ✓

And if $x=36$, then LHS $=3-\left|\frac{1}{2}\cdot 36-5\right|=3-\left|18-5\right|=3-\left|13\right|=3-13=-10=$ RHS ✓. So the solutions are $\boxed{-16 \text{ and } 36}$.

c) Again, we first solve for the expression with the absolute value sign.

$$\begin{aligned} 8+3\left|5x-4\right| &= 2 && \text{subtract } 8 \\ 3\left|5x-4\right| &= -6 && \text{divide by } 3 \\ \left|5x-4\right| &= -2 \end{aligned}$$

The absolute value of no number is negative. Therefore, we can conclude that this equation has $\boxed{\text{no solution}}$. Some students might solve for $5x-4=\pm 2$. In this case, it is essential that we check.

Some equatons with absolute value are more complicated.

Example 4. Solve each of the given equations.

a) $|4x-5| = |x+1|$ b) $|2x-3| = -x-4$

Solution: a) How can the absolute value of two quantities be equal? There are only two ways: if the quantities are equal or if they are opposites of each other. This way we will, again, trade one equation to two linear.

$$|4x-5| = |x+1| \quad \Longrightarrow \quad 4x-5 = x+1 \quad \text{or} \quad 4x-5 = -(x+1)$$

We solve both linear equations for x.

$4x-5 = x+1$	subtract x		$4x-5 = -x-1$	add x
$3x-5 = 1$	add 5		$5x-5 = -1$	add 5
$3x = 6$	divide by 3		$5x = 4$	divide by 5
$x = 2$			$x = \frac{4}{5}$	

This equation seems too has two solutions. We check:

if $x = 2$, then LHS $= |4 \cdot 2 - 5| = |8-5| = |3| = 3$ and

RHS $= |2+1| = |3| = 3$ ✓

If $x = \frac{4}{5}$, then LHS $= \left|4 \cdot \frac{4}{5} - 5\right| = \left|\frac{16}{5} - \frac{25}{5}\right| = \left|-\frac{9}{5}\right| = \frac{9}{5}$ and

RHS $= \left|\frac{4}{5} + 1\right| = \left|\frac{4}{5} + \frac{5}{5}\right| = \left|\frac{9}{5}\right| = \frac{9}{5}$ ✓ So the solutions are $\boxed{2 \text{ and } \frac{4}{5}}$.

b) While this equation looks similar to the previous one, the slight difference will have significant consequences.

$$|2x-3| = -x-4 \quad \Longrightarrow \quad 2x-3 = -x-4 \quad \text{or} \quad 2x-3 = -(-x-4)$$

We solve both linear equations for x.

$2x-3 = -x-4$	add x		$2x-3 = -(-x-4)$	simplify
$3x-3 = -4$	add 3		$2x-3 = x+4$	subtract x
$3x = -1$	divide by 3		$x-3 = 4$	add 3
$x = -\frac{1}{3}$			$x = 7$	

This equation seems too has two solutions. We check: $|2x-3| = -x-4$

if $x = -\frac{1}{3}$, then LHS $= \left|2\left(-\frac{1}{3}\right) - 3\right| = \left|-\frac{2}{3} - \frac{9}{3}\right| = \left|-\frac{11}{3}\right| = \frac{11}{3}$ and

RHS $= -\left(-\frac{1}{3}\right) - 4 = \frac{1}{3} - \frac{12}{3} = -\frac{11}{3}$ so $-\frac{1}{3}$ is NOT a solution!

And if $x = 7$, then LHS $= |2 \cdot 7 - 3| = |14-3| = |11| = 11$ and

RHS $= -(7+4) = -11$ Thus, 7 is also not a solution! So this equation has $\boxed{\text{no solution}}$.

Practice Problems

Solve each of the given equations.

1. $|2x+7| = 1$
2. $\frac{1}{2}|x-5| = 6$
3. $2 - |3x+8| = 10$
4. $10 - |3x+8| = 2$
5. $2 - \frac{1}{5}|2x+17| = -3$
6. $|3x-6| + 4 = 7$
7. $|3x-6| = 6$
8. $|3x-6| = 0$
9. $3 + \left|\frac{1}{2}x - 3\right| = 7$
10. $3 - \left|\frac{1}{2}x - 3\right| = 7$
11. $2 - 3|2x-5| = -10$
12. $|3x-1| = 5$
13. $|3x-5| = 1$
14. $|4x-7| - 7 = 0$
15. $|4x-7| - 7 = 7$
16. $|x+1| = |2x-7|$
17. $|5x-2| = |-x+8|$
18. $|x-5| = |x+3|$
19. $x - 5 = |2x+1|$
20. $|x-5| = 2x+1$

Answers

1. $-4,-3$ 2. $-7,17$ 3. no solution 4. $-\frac{16}{3},0$ 5. $-21,4$ 6. $1,3$ 7. $0,4$ 8. 2

9. $-2,14$ 10. no solution 11. $\frac{1}{2},\frac{9}{2}$ 12. $-\frac{4}{3},2$ 13. $\frac{4}{3},2$ 14. $0,\frac{7}{2}$ 15. $-\frac{7}{4},\frac{21}{4}$

16. $2,8$ 17. $-\frac{3}{2},\frac{5}{3}$ 18. 1 19. no solution 20. $\frac{4}{3}$