Part 1 - What is Combinatorics?

Combinatorics is a branch of mathematics that studies counting techniques. Combinatorial questions usually have non-negative integer answers. No negative numbers, no fractions. To illustrate the study of combinatorics, we start with a few examples of famous questions from combinatorics. A warning to the reader: do not expect that we will immediately know how to solve all of these problems.

Question 1. There are 100 people in a large conference room. If everyone greeted everyone else in the room with a handshake, how many handshakes would take place?

Question 2. There are 50 married couple in a large conference room. If everyone greeted everyone else in the room - except for their own spouse - with a handshake, how many handshakes would take place?

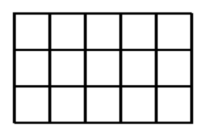
Question 3. We have a group of 9 students, 3 females and 6 males. In how many different ways can we form three groups of three students?

Question 4. We have a group of 9 students, 3 females and 6 males. In how many different ways can we form three groups of three students where each group contains exactly one female?

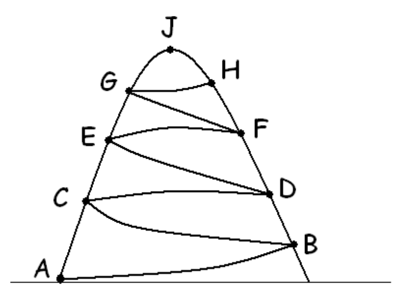
Question 5. We have 20 letters to mail to different addresses, with different content. We also have 20 addressed envelopes. Somehow the letters get all mixed up. If we place 1 letter into each envlope, in how many different ways can we do that?

Question 6. In Illinois, the Mega Lottery's rules are that we randomly select six numbers out of $\{1, 2, 3, ..., 45\}$. If we wanted to fill out a lottery ticket for every possible outcome, how many tickets do we need to purchase?

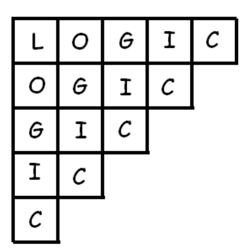
Question 7. How many rectangles are there on the picture? How many squares?



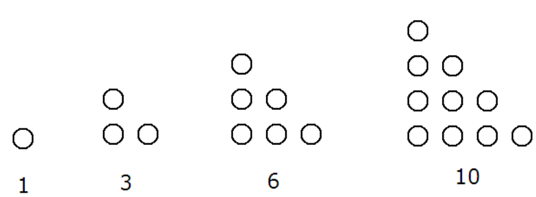
Question 8. As the picture shows, a mountain has two different tracks leading to its top. There is a mild, comfortable path, and also a steep, difficult track. People can choose at each intersection which path to take. Given that walking downward is not allowed, how many different pathways are there from point A to point J? (For example, ABDFGHJ and ABCEFHJ are two different pathways.)



Question 9. In how many different ways can we spell the word LOGIC based on the picture? Assume that we start in the upper left corner and only step to the right or down.



Question 10. The first few triangular numbers are 1, 3, 6, and 10. What is the 100th triangular number?



Part 2 - The First Counting Technique: Systematic Listing

The first technique is **systematic listing**. We want to know how many cases are there? We write them all down and count them. We have to be systematic to ensure that the count is correct - the two evils to avoid are skipping something and counting something more than once. For this reason, the gretest challenge in the problem is to develop a system or language for the counting. The rest is then easy.

We should never, never feel 'too good' for systematic listing. In many cases, when faced with a combinatorial problem and there is no immediate insight, we simply start with systematic listing. As we do that, we often get the key idea(s) that help us selecting and setting the more complex counting methods.

Example 1. List all different $3-$digit numbers that can be formed using only the digits 1, 2, 3, and 4.

Solution: We are systematic if we simply insist on counting the numbers in an increasing order. The smallest number we can create is 111. To avoid skipping them, we change just the end of the number: we 'roll up' to 112. Then 113. Then 114.

$$111, 112, 113, 114$$

The next number can not be 115 because we can not use the digit 5. So what is the smallest next number? The answer is: 121. We list all possible numbers with the first digit is 1 and the second digit is 2.

111	121
112	122
113	123
114	124

Then we roll that 2 in the tenth place up to 3 and then to 4.

111	121	131	141
112	122	132	142
113	123	133	143
114	124	134	144

What do we have at this point? Well, we found 16 such numbers, but this is the complete list of all possible numbers that sart with 1. We will have a similar table for numbers starting with 2, 3, and 4, giving us

111	121	131	141
112	122	132	142
113	123	133	143
114	124	134	144

16 numbers, starting with 1

211	221	231	241
212	222	232	242
213	223	233	243
214	224	234	244

16 numbers, starting with 2

311	321	331	341
312	322	332	342
313	323	333	343
314	324	334	344

16 numbers, starting with 3

411	421	431	441
412	422	432	442
413	423	433	443
414	424	434	444

16 numbers, starting with 4

And so we listed all such numbers. Please notice that without a system to follow, (in this case we listed the numbers in an increasing order) gathering these 64 numbers would have been much more difficult.

Example 2. List all different 3−digit numbers with three different digits that can be formed using the digits 1, 2, 3, and 4.

Solution: This problem is similar to the previous one. The only difference is: this time repetition of digits is not allowed. So numbers such as 133 or 141 needed to be counted before, in this case those can not be counted. We can ensure that we are systematic if, again, just list these numbers in an increasing order. The first (i.e. smallest) few such numbers are

123	132	142
124	134	143

and we already listed all such numbers that start with the digit 1. Clearly, this list will be shorter than the previous problem's list. We will have a similar table for numbers starting with 2, 3, and 4, giving us

6 numbers, starting with 1			6 numbers, starting with 2			6 numbers, starting with 3			6 numbers, starting with 4		
123	132	142	213	231	241	312	321	341	412	421	431
124	134	143	214	234	243	314	324	342	413	423	432

And so we listed all such numbers. Please notice that without a system to follow, (in this case we listed the numbers in an increasing order) gathering these 24 numbers would have been much more difficult.

Example 3. In a miniature lottery game, we randomly pull three numbers out of $\{1,2,3,4,5,6,7,8\}$. List all possible outcomes.

Solution: It doesn't matter in what order the numbers come in, at the end, we organize and announce the winning numbers in an increasing order. So, we will do the same. Notice the spatial arrangement helping in counting.

1,2,3					
1,2,4	1,3,4				
1,2,5	1,3,5	1,4,5			
1,2,6	1,3,6	1,4,6	1,5,6		
1,2,7	1,3,7	1,4,7	1,5,7	1,6,7	
1,2,8	1,3,8	1,4,8	1,5,8	1,6,8	1,7,8

These are only the outcomes where the smallest winning number is 1. What if the smallest winning number is 2?

2,3,4				
2,3,5	2,4,5			
2,3,6	2,4,6	2,5,6		
2,3,7	2,4,7	2,5,7	2,6,7	
2,3,8	2,4,8	2,5,8	2,6,8	2,7,8

These are all the possible outcomes in which the smallest number is 2. What if the smallest number is 3?

3,4,5			
3,4,6	3,5,6		
3,4,7	3,5,7	3,6,7	
3,4,8	3,5,8	3,6,8	3,7,8

As we proceed, we find less and less outcomes with higher and higher numbers as their smallest number.

4,5,6		
4,5,7	4,6,7	
4,5,8	4,6,8	4,7,8

5,6,7	
5,6,8	5,7,8

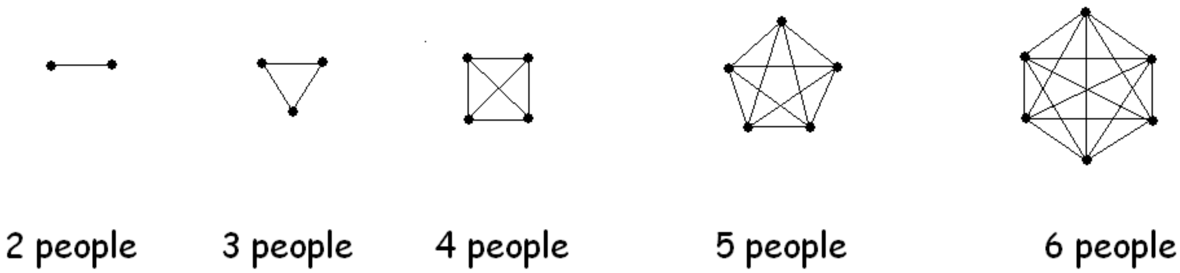
6,7,8

So we listed all 56 possible outcomes.

Part 3 - The Handshake Problem

Example 4. There are 100 people in a large conference room. If everyone greeted everyone else in the room with a handshake, how many handshakes would take place?

The problem is worth exploring. We can investigate the question using smaller crowds and systematic listing. However, this method eventually becomes too laborious for numbers such as 100. One representation could be using pictures representing people with points and the handshakes with a line connecting them.



So in case of 2 people, there is just 1 handshake. With 3 people, there are 3 handshakes. And so on, but we notice that counting the line segments is becoming more and more difficult as the numbers increase. Such a picture is called a graph, and graph theory is a branch of combinatorics that studies the properties of such geometric objects.

So we should look for another way to systematically list them. Let us label the people by numbers 1,2,3,4, and 5. Then one handshake between persons 2 and 5 could be denoted by $2,5$ or even 25. We do not need to list $2,5$ and $5,2$ as they refer to the same handshake. So, systematic listing of handshakes for 4 people would be

for 4 people:			
	1,2		
	1,3	2,3	
	1,4	2,4	3,4
	6 handshakes		

for 5 people:				
	1,2			
	1,3	2,3		
	1,4	2,4	3,4	
	1,5	2,5	3,5	4,5
	10 handshakes			

for 6 people:					
	1,2				
	1,3	2,3			
	1,4	2,4	3,4		
	1,5	2,5	3,5	4,5	
	1,6	2,6	3,6	4,6	5,6
	15 handshakes				

Systematic listing gets us only a slick technique for counting, but the numbers we obtained don't seem to follow an easy pattern.

people	handshakes
2	1
3	3
4	6
5	10
6	21

Attention! Never feel 'to good' for systematic listing. Even if you do not get immediate answers to your current question, there is much to discover, (even in this particular case) and those patterns and connections reveal themselves as we apply and improve systematic listing.

The craft becomes art when we move beyond listing. Many combinatorial ideas are based on a reorganization of how we count; on creating a new choregraphy.

Imagine the 100 people in the room. You are an extra person, from the media. You interview every single person in the room and ask everyone how many times they shook hands today in the room. Every single person would reply the same number: 99 times. So, we have a 100 answers, each of them 99. Does that mean that we have $100 \cdot 99 = 9900$ many handshakes? Well, we are not sure. One importance of listing is that they provide the correct answers for the first smallest few cases. Therefore, we can use them to test our idea.

Suppose we have three people in the room. We interview all three of them. They all claim to have shaken two hands. $3 \cdot 2 = 6$ but the correct number (we know from listing) should be 3.

Suppose we have four people in the room. We interview all four of them. They all claim to have shaken three hands. $4 \cdot 3 = 12$ but the correct number (we know from listing) should be 6.

Suppose we have five people in the room. We interview all five of them. They all claim to have shaken four hands. $5 \cdot 4 = 20$ but the correct number (we know from listing) should be 10.

We notice now that our formula produces exactly twice the correct answer. Why would this be? Our method of multiplying the number of people by the handshakes results in counting **each** handshake **twice**. So, we adjust for this error and divide our number by 2. Let us first see if this idea produces the right results.

2 people	$\frac{2 \cdot 1}{2} = 1 \quad \checkmark$	5 people	$\frac{5 \cdot 4}{2} = 10 \quad \checkmark$
3 people	$\frac{3 \cdot 2}{2} = 3 \quad \checkmark$	6 people	$\frac{6 \cdot 5}{2} = 15 \quad \checkmark$
4 people	$\frac{4 \cdot 3}{2} = 6 \quad \checkmark$		

and so we have found the correct counting method that can now be applied to 100.

$$\text{number of handshakes} = \frac{100 \cdot 99}{2} = 4950$$

The development of such choreographies or counting arguments is a central technique in combinatorics.

Practice Problems

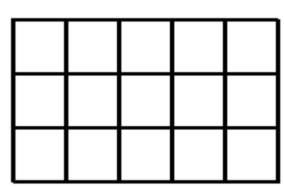
1. There are 50 married couple in a large conference room. If everyone greeted everyone else in the room - except for their own spouse - with a handshake, how many handshakes would take place?

2. There are 50 married couple in a large conference room. If every man greeted every other man in the room with a handshake and also every woman greeted every other woman in the room with a handshake, how many handshakes would take place?

3. List all $4-$element subsets of the set $\{1,2,3,4,5,6\}$.

4. List all subsets of $A = \{1,2,3,4,5\}$

5. How many rectangles are there on the picture shown? How many squares?



Answers

1. 4900 2. 2450

3.

$\{1,2,3,4\}$
$\{1,2,3,5\}$ $\{1,2,4,5\}$
$\{1,2,3,6\}$ $\{1,2,4,6\}$ $\{1,2,5,6\}$ smallest two numbers are 1 and 2

$\{1,3,4,5\}$ $\{1,3,5,6\}$ smallest two numbers are 1 and 3
$\{1,3,4,6\}$

$\{1,4,5,6\}$ smallest two numbers are 1 and 4

$\{2,3,4,5\}$
$\{2,3,4,6\}$ $\{2,3,5,6\}$ smallest numbers are 2 and 3

$\{2,4,5,6\}$ smallest numbers are 2 and 4

$\{3,4,5,6\}$ smallest numbers are 3 and 4

4. There are 32 subsets, listed below.

0-element subsets: $\varnothing$ 1 subset
1-element subsets: $\{1\},\{2\},\{3\},\{4\},\{5\}$ 5 subsets

2-element subsets: 10 subsets
$\{1,2\}$
$\{1,3\}$ $\{2,3\}$
$\{1,4\}$ $\{2,4\}$ $\{3,4\}$
$\{1,5\}$ $\{2,5\}$ $\{3,5\}$ $\{4,5\}$

3-element subsets: 10 subsets
$\{1,2,3\}$
$\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$
$\{1,2,5\}$ $\{1,3,5\}$ $\{1,4,5\}$ $\{2,3,5\}$ $\{2,4,5\}$ $\{3,4,5\}$

4-element subsets: $\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}$ 5 subsets
5-element subsets: $\{1,2,3,4,5\}$ 1subsets

5. rectangles: 90 squares: 26

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