One important achievement of our studies is the ability to solve quadratic equations with any real coefficieints. In what follows, we will master this important skill. As a matter of fact, we can apply the method of completing the square to any quadratic expression. The steps are still the same - it's just that each step takes a bit more work.

Recall that a **perfect square** is the square of in integer. For example, 4, 25, or 49 are perfect squares.

Recall that if N is a non-negative number, then $\sqrt{N}$ is the non-negative number x such that $x^2 = N$. For example, $\sqrt{7}$ is the non-negative number, that, when we square, we get 7. In short, $\left(\sqrt{7}\right)^2 = 7$. In what follows, we start interpreting numbers that are not perfect squares as squares anyway. For example, we can interpret 7 as $\left(\sqrt{7}\right)^2$ or 5 as $\left(\sqrt{5}\right)^2$. This will be useful when we want to apply the difference of squares theorem.

Example 1. Solve the equation $x^2 + 13 = 8x$ over the real numbers. Make sure to check your solutions.

Solution: Since the equation is quadratic, we reduce one side to zero and factor the other side by completing the square.

$$
\begin{aligned}
x^2 + 13 &= 8x \\
x^2 - 8x + 13 &= 0 && (x-4)^2 = x^2 - 8x + 16 \quad \text{so we smuggle in 16} \\
\underbrace{x^2 - 8x + 16} - 16 + 13 &= 0 \\
(x-4)^2 - 3 &= 0 && 3 = \left(\sqrt{3}\right)^2 \\
(x-4)^2 - \left(\sqrt{3}\right)^2 &= 0 && \text{factor via difference of squares theorem} \\
\left(x - 4 + \sqrt{3}\right)\left(x - 4 - \sqrt{3}\right) &= 0
\end{aligned}
$$

We apply the zero product rule, and solve for the zeroes of each linear factor.

$$
\begin{aligned}
x - 4 + \sqrt{3} &= 0 & \text{or} \qquad & x - 4 - \sqrt{3} = 0 \\
x + \sqrt{3} &= 4 & & x - \sqrt{3} = 4 \\
x_1 &= 4 - \sqrt{3} & & x_2 = 4 + \sqrt{3}
\end{aligned}
$$

$$x_1 = 4 - \sqrt{3} \qquad x_2 = 4 + \sqrt{3} \qquad \text{or in shorter notation: } x_{1,2} = 4 \pm \sqrt{3}$$

We check: if $x = 4 - \sqrt{3}$, then

$$
\begin{aligned}
\text{LHS} &= \left(4 - \sqrt{3}\right)^2 + 13 = 16 - 4\sqrt{3} - 4\sqrt{3} + 3 + 13 = 32 - 8\sqrt{3} \\
\text{RHS} &= 8\left(4 - \sqrt{3}\right) = 32 - 8\sqrt{3}
\end{aligned}
$$

If $x = 4 + \sqrt{3}$, then

$$
\begin{aligned}
\text{LHS} &= \left(4 + \sqrt{3}\right)^2 + 13 = 16 + 4\sqrt{3} + 4\sqrt{3} + 3 + 13 = 32 + 8\sqrt{3} \\
\text{RHS} &= 8\left(4 + \sqrt{3}\right) = 32 + 8\sqrt{3}
\end{aligned}
$$

Thus our solution, $\boxed{4 + \sqrt{3} \text{ and } 4 - \sqrt{3}}$ is correct.

Our solutions are irrational numbers. When a quadratic equation has irrational solutions, all other methods break down, and only completing the square works. If the reader is familiar with methods such as grouping or the AC-method, or trial and error, try solving $x^2 + 13 = 8x$ using any of those methods!

Being able to handle irrational solutions is a great advantage of the method of completing the square.

Please also note that in case of irrational solutions, writing the exact value necessitates the use of radicals. Recall that the exact value of an irrational number can not be captured as a fraction or as a terminating or repeating decimal, so radical form is our only option.

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Example 2. Solve the equation $x^2 = 4x + 1$ over the real numbers. Make sure to check your solutions.

Solution : We complete the square.

$$\begin{aligned} x^2 &= 4x+1 \\ x^2-4x-1 &= 0 \\ \underbrace{x^2-4x+4}-4-1 &= 0 \\ (x-2)^2-5 &= 0 \\ (x-2)^2-\left(\sqrt{5}\right)^2 &= 0 \\ \left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right) &= 0 \end{aligned}$$

$$(x-2)^2 = x^2-4x+4 \qquad 5 = \left(\sqrt{5}\right)^2$$

We apply the zero product rule to the product of the two linear factors.

$$\begin{aligned} x-2+\sqrt{5} &= 0 \\ x+\sqrt{5} &= 2 \\ x_1 &= 2-\sqrt{5} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x-2-\sqrt{5} &= 0 \\ x-\sqrt{5} &= 2 \\ x_2 &= 2+\sqrt{5} \end{aligned}$$

$$x_1 = 2-\sqrt{5} \quad \text{and} \quad x_2 = 2+\sqrt{5} \quad \text{or in shorter notation: } x_{1,2} = 2 \pm \sqrt{5}$$

We check: if $x = 2-\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2-\sqrt{5}\right)^2 = 4-2\sqrt{5}-2\sqrt{5}+5 = 9-4\sqrt{5} \\ \text{RHS} &= 4\left(2-\sqrt{5}\right)+1 = 8-4\sqrt{5}+1 = 9-4\sqrt{5} \end{aligned}$$

If $x = 2+\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2+\sqrt{5}\right)^2 = 4+2\sqrt{5}+2\sqrt{5}+5 = 9+4\sqrt{5} \\ \text{RHS} &= 4\left(2+\sqrt{5}\right)+1 = 8+4\sqrt{5}+1 = 9+4\sqrt{5} \end{aligned}$$

Thus our solution, $\boxed{2+\sqrt{5} \text{ and } 2-\sqrt{5}}$ is correct.

Example 3. Solve the equation $2x^2 + 10x + 9 = 0$ over the real numbers. Make sure to check your solutions.

Solution: We will factor by completing the square. We first factor out the leading coefficient.

$$\begin{aligned} 2x^2+10x+9 &= 0 \\ 2\left(x^2+5x+\frac{9}{2}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\frac{5}{2}$. Thus the complete square is $\left(x+\frac{5}{2}\right)^2$

$$\left(x+\frac{5}{2}\right)^2 = \left(x+\frac{5}{2}\right)\left(x+\frac{5}{2}\right) = x^2+\frac{5}{2}x+\frac{5}{2}x+\frac{25}{4} = x^2+5x+\frac{25}{4}$$

So we smuggle in $\frac{25}{4}$.

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$$2\left(\underbrace{x^2+5x+\frac{25}{4}}-\frac{25}{4}+\frac{9}{2}\right) = 0 \qquad \frac{9}{2}=\frac{18}{4}$$

$$2\left(\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{18}{4}\right) = 0$$

$$2\left(\left(x+\frac{5}{2}\right)^2-\frac{7}{4}\right) = 0$$

We next apply the difference of squares theorem. $\frac{7}{4}$ is the square of a number - its own square root.

$$2\left(\left(x+\frac{5}{2}\right)^2-\frac{7}{4}\right) = 0 \qquad \frac{7}{4}=\left(\sqrt{\frac{7}{4}}\right)^2=\left(\frac{\sqrt{7}}{\sqrt{4}}\right)^2=\left(\frac{\sqrt{7}}{2}\right)^2$$

$$2\left(\left(x+\frac{5}{2}\right)^2-\left(\frac{\sqrt{7}}{2}\right)^2\right) = 0$$

$$2\left(x+\frac{5}{2}+\frac{\sqrt{7}}{2}\right)\left(x+\frac{5}{2}-\frac{\sqrt{7}}{2}\right) = 0$$

We solve for the zeroes of both linear factors:

$$x+\frac{5}{2}+\frac{\sqrt{7}}{2} = 0 \qquad \text{or} \qquad x+\frac{5}{2}-\frac{\sqrt{7}}{2}=0$$

$$x+\frac{\sqrt{7}}{2} = -\frac{5}{2} \qquad\qquad x-\frac{\sqrt{7}}{2}=-\frac{5}{2}$$

$$x_1 = -\frac{5}{2}-\frac{\sqrt{7}}{2}=\frac{-5-\sqrt{7}}{2} \qquad\qquad x_2=-\frac{5}{2}+\frac{\sqrt{7}}{2}=\frac{-5+\sqrt{7}}{2}$$

So the answer is $\boxed{x_1=\frac{-5-\sqrt{7}}{2} \text{ and } x_2=\frac{-5+\sqrt{7}}{2}}$ or, in a more compact form, $x_{1,2}=\frac{-5\pm\sqrt{7}}{2}$.

We check: if $x=\frac{-5-\sqrt{7}}{2}$, then

$$\begin{aligned}
\text{LHS} &= 2x^2+10x+9=2\left(\frac{-5-\sqrt{7}}{2}\right)^2+10\left(\frac{-5-\sqrt{7}}{2}\right)+9=\frac{2}{1}\cdot\frac{\left(-5-\sqrt{7}\right)^2}{2^2}+\frac{10}{1}\cdot\frac{-5-\sqrt{7}}{2}+9\\
&= \frac{2\left(-5-\sqrt{7}\right)^2}{4}+\frac{10\left(-5-\sqrt{7}\right)}{2}+\frac{9}{1}=\frac{\left(-5-\sqrt{7}\right)^2}{2}+\frac{10\left(-5-\sqrt{7}\right)}{2}+\frac{18}{2}=\\
&= \frac{25+7+10\sqrt{7}-50-10\sqrt{7}+18}{2}=\frac{32+10\sqrt{7}-50-10\sqrt{7}+18}{2}=0=\text{RHS}
\end{aligned}$$

and if $x=\frac{-5+\sqrt{7}}{2}$, then

$$\begin{aligned}
\text{LHS} &= 2x^2+10x+9=2\left(\frac{-5+\sqrt{7}}{2}\right)^2+10\left(\frac{-5+\sqrt{7}}{2}\right)+9=\frac{2}{1}\cdot\frac{\left(-5+\sqrt{7}\right)^2}{2^2}+\frac{10}{1}\cdot\frac{-5+\sqrt{7}}{2}+9\\
&= \frac{2\left(-5+\sqrt{7}\right)^2}{4}+\frac{10\left(-5+\sqrt{7}\right)}{2}+\frac{9}{1}=\frac{\left(-5+\sqrt{7}\right)^2}{2}+\frac{10\left(-5+\sqrt{7}\right)}{2}+\frac{18}{2}=\\
&= \frac{25+7-10\sqrt{7}-50+10\sqrt{7}+18}{2}=\frac{32-10\sqrt{7}-50+10\sqrt{7}+18}{2}=0=\text{RHS}
\end{aligned}$$

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Thus our solution, $\dfrac{-5-\sqrt{7}}{2}$ and $\dfrac{-5-\sqrt{7}}{2}$ is correct.

Example 4. Is there a real number that is exactly one greater than its own reciprocal?

Solution: Let us denote this number by x. The reciprocal of x is then $\dfrac{1}{x}$. We set up and solve the equation comparing the number to its own reciprocal.

$$x = \frac{1}{x} + 1$$

We clear the denominator if we multiply both sides of the equation by x. Then we obtain a quadratic equation that we will solve by completing the square.

$$\begin{aligned}
x &= \frac{1}{x}+1 && \text{multiply by } x \\
x^2 &= 1+x \\
x^2-x-1 &= 0 && \left(x-\frac{1}{2}\right)^2 = x^2-x+\frac{1}{4} \\
\underbrace{x^2-x+\frac{1}{4}}-\frac{1}{4}-1 &= 0 \\
\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-\frac{4}{4} &= 0 \\
\left(x-\frac{1}{2}\right)^2-\frac{5}{4} &= 0 && \frac{5}{4} = \left(\sqrt{\frac{5}{4}}\right)^2 = \left(\frac{\sqrt{5}}{\sqrt{4}}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 \\
\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2 &= 0 \\
\left(x-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x-\frac{1}{2}-\frac{\sqrt{5}}{2}\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\text{Either } x-\frac{1}{2}+\frac{\sqrt{5}}{2} &= 0 && \text{or} && x-\frac{1}{2}-\frac{\sqrt{5}}{2} = 0 \\
x+\frac{\sqrt{5}}{2} &= \frac{1}{2} && && x-\frac{\sqrt{5}}{2} = \frac{1}{2} \\
x_1 &= \frac{1}{2}-\frac{\sqrt{5}}{2} && && x_2 = \frac{1}{2}+\frac{\sqrt{5}}{2}
\end{aligned}$$

The greater number, $x_2 = \dfrac{1+\sqrt{5}}{2} \approx 1.618\,034$ is a famous number. It is called the golden mean or golden ratio. $\dfrac{1+\sqrt{5}}{2}$ is often denoted by φ and has very interesting properties. For example, it is one greater than its own reciprocal.

$$\frac{1}{\varphi}+1 = \frac{2}{1+\sqrt{5}}+1 = \frac{2}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1}+1 = \frac{2\left(\sqrt{5}-1\right)}{4}+1 = \frac{\sqrt{5}-1}{2}+\frac{2}{2} = \frac{\sqrt{5}-1+2}{2} = \frac{\sqrt{5}+1}{2} = \varphi$$

The smaller number also works. Let us denote $\dfrac{1-\sqrt{5}}{2}$ by ψ.

$$\frac{1}{\psi}+1 = \frac{2}{1-\sqrt{5}}+1 = \frac{2}{1-\sqrt{5}}\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}}+1 = \frac{2\left(1+\sqrt{5}\right)}{-4}+1 = \frac{-1\left(1+\sqrt{5}\right)}{2}+\frac{2}{2} = \frac{-1-\sqrt{5}+2}{2} = \frac{1-\sqrt{5}}{2} = \psi$$

Sample Problems

1. Solve the equation $3x^2 - 5x - 7 = 0$ over the real numbers. Make sure to check your solutions.
2. Find all numbers that are exactly two less than their own reciprocal.

Practice Problems

Solve each of the following equations over the real numbers. Make sure to check your solutions.

1. $x^2 - 13 = 6x$
2. $x^2 = 10x + 3$
3. $x^2 + x - 1 = 0$
4. $3x^2 - 11x + 5 = 0$
5. $-2x^2 + 8x - 29 = 0$
6. $5x^2 - 4x = 20$
7. $x^2 + 8x - 10 = 0$
8. $3x^2 + x - 1 = 0$
9. $x^2 + x + 1 = 0$

Answers

Sample Problems

1. $\dfrac{5-\sqrt{109}}{6}$ and $\dfrac{5+\sqrt{109}}{6}$ 2. $\sqrt{2}-1$ and $-1-\sqrt{2}$

Practice Problems

1. $3\pm\sqrt{22}$ 2. $5\pm2\sqrt{7}$ 3. $\dfrac{-1\pm\sqrt{5}}{2}$ 4. $\dfrac{11\pm\sqrt{61}}{6}$ 5. no real solution

6. $\dfrac{2\pm2\sqrt{26}}{5}$ 7. $-4\pm\sqrt{26}$ 8. $\dfrac{-1\pm\sqrt{13}}{6}$ 9. no real solution

Sample Problems Solutions

1. Solve the equation $3x^2-5x-7=0$ over the real numbers. Make sure to check your solutions.

   Solution: We factor out the leading coefficient and factor by completing the square.

$$\begin{aligned} 3x^2-5x-7 &= 0 \\ 3\left(x^2-\frac{5}{3}x-\frac{7}{3}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\dfrac{5}{3}\div 2=\dfrac{5}{3}\cdot\dfrac{1}{2}=\dfrac{5}{6}$. Thus the complete square is $\left(x-\dfrac{5}{6}\right)^2$

$$\left(x-\frac{5}{6}\right)^2=\left(x-\frac{5}{6}\right)\left(x-\frac{5}{6}\right)=x^2-\frac{5}{6}x-\frac{5}{6}x+\frac{25}{36}=x^2-\frac{5}{3}x+\frac{25}{36}$$ So we smuggle in $\dfrac{25}{36}$

$$\begin{aligned} 3\left(\underbrace{x^2-\frac{5}{3}x+\frac{25}{36}}-\frac{25}{36}-\frac{7}{3}\right) &= 0 \\ 3\left(\left(x-\frac{5}{6}\right)^2-\frac{25}{36}-\frac{7\cdot 12}{3\cdot 12}\right) &= 0 \\ 3\left(\left(x-\frac{5}{6}\right)^2-\frac{25}{36}-\frac{84}{36}\right) &= 0 \\ 3\left(\left(x-\frac{5}{6}\right)^2-\frac{109}{36}\right) &= 0 \end{aligned}$$

We next apply the difference of squares theorem. $\dfrac{109}{36}$ is the square of a number - its own square root.

$$\frac{109}{36}=\left(\sqrt{\frac{109}{36}}\right)^2=\left(\frac{\sqrt{109}}{\sqrt{36}}\right)^2=\left(\frac{\sqrt{109}}{6}\right)^2$$

$$3\left(\left(x-\frac{5}{6}\right)^2-\left(\frac{\sqrt{109}}{6}\right)^2\right) = 0$$

$$3\left(x-\frac{5}{6}+\frac{\sqrt{109}}{6}\right)\left(x-\frac{5}{6}-\frac{\sqrt{109}}{6}\right) = 0$$

We solve for the zeroes of both linear factors:

$$x-\frac{5}{6}+\frac{\sqrt{109}}{6} = 0 \qquad \text{or} \qquad x-\frac{5}{6}-\frac{\sqrt{109}}{6}=0$$

$$x+\frac{\sqrt{109}}{6} = \frac{5}{6} \qquad\qquad x-\frac{\sqrt{109}}{6}=\frac{5}{6}$$

$$x_1 = \frac{5}{6}-\frac{\sqrt{109}}{6}=\boxed{\frac{5-\sqrt{109}}{6}} \qquad\qquad x_2=\frac{5}{6}+\frac{\sqrt{109}}{6}=\boxed{\frac{5+\sqrt{109}}{6}}$$

We check: if $x=\dfrac{5-\sqrt{109}}{6}$, then

$$\begin{aligned}
\text{LHS} &= 3x^2-5x-7=3\left(\frac{5-\sqrt{109}}{6}\right)^2-5\left(\frac{5-\sqrt{109}}{6}\right)-7\\
&= \frac{3}{1}\cdot\frac{\left(5-\sqrt{109}\right)^2}{36}-\frac{5}{1}\left(\frac{5-\sqrt{109}}{6}\right)-7=\\
&= \frac{3\left(25-5\sqrt{109}-5\sqrt{109}+109\right)}{36}-\frac{5\left(5-\sqrt{109}\right)}{6}-7=\frac{134-10\sqrt{109}}{12}-\frac{25-5\sqrt{109}}{6}-7\\
&= \frac{134-10\sqrt{109}}{12}-\frac{2\left(25-5\sqrt{109}\right)}{12}-7=\frac{134-10\sqrt{109}-2\left(25-5\sqrt{109}\right)}{12}-7\\
&= \frac{134-10\sqrt{109}-50+10\sqrt{109}}{12}-7=\frac{84}{12}-7=7-7=0=\text{RHS}
\end{aligned}$$

and if $x=\dfrac{5+\sqrt{109}}{6}$, then

$$\begin{aligned}
\text{LHS} &= 3x^2-5x-7=3\left(\frac{5+\sqrt{109}}{6}\right)^2-5\left(\frac{5+\sqrt{109}}{6}\right)-7\\
&= \frac{3}{1}\cdot\frac{\left(5+\sqrt{109}\right)^2}{36}-\frac{5}{1}\left(\frac{5+\sqrt{109}}{6}\right)-7\\
&= \frac{3\left(25+5\sqrt{109}+5\sqrt{109}+109\right)}{36}-\frac{5\left(5+\sqrt{109}\right)}{6}-7=\frac{134+10\sqrt{109}}{12}-\frac{25+5\sqrt{109}}{6}-7\\
&= \frac{134+10\sqrt{109}}{12}-\frac{2\left(25+5\sqrt{109}\right)}{12}-7=\frac{134+10\sqrt{109}-2\left(25+5\sqrt{109}\right)}{12}-7\\
&= \frac{134+10\sqrt{109}-50-10\sqrt{109}}{12}-7=\frac{84}{12}-7=7-7=0=\text{RHS}
\end{aligned}$$

Thus our solution, $\dfrac{5+\sqrt{109}}{6}$ and $\dfrac{5-\sqrt{109}}{6}$ is correct.

2. Find all numbers that are exactly two less than their own reciprocal.

Solution: Let us denote this number by x. Then its reciprocal is $\dfrac{1}{x}$ and

$$\begin{aligned}
x &= \frac{1}{x} - 2 && \text{multiply by } x \\
x^2 &= 1 - 2x \\
x^2 + 2x - 1 &= 0 && (x+1)^2 = x^2 - 2x + 1 \\
\underbrace{x^2 + 2x + 1} - 1 - 1 &= 0 \\
(x+1)^2 - 2 &= 0 \\
(x+1)^2 - \left(\sqrt{2}\right)^2 &= 0 \\
(x+1)^2 - \left(\sqrt{2}\right)^2 &= 0 \\
\left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right) &= 0
\end{aligned}$$

Either $x + 1 + \sqrt{2} = 0$ or $x + 1 - \sqrt{2} = 0$

$x + \sqrt{2} = -1$ $\qquad$ $x - \sqrt{2} = -1$

$x_1 = -1 - \sqrt{2}$ $\qquad$ $x_2 = -1 + \sqrt{2}$

We check: If $x = -1 - \sqrt{2}$, then

$$\begin{aligned}
\frac{1}{x} - 2 &= \frac{1}{-1-\sqrt{2}} - 2 = \frac{1(-1)}{\left(-1-\sqrt{2}\right)(-1)} - 2 = \frac{-1}{\sqrt{2}+1} - 2 = \frac{-1}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} - 2 \\
&= \frac{-1\left(\sqrt{2}-1\right)}{2-1} - 2 = \frac{-\sqrt{2}+1}{1} - 2 = -\sqrt{2} + 1 + 2 = -\sqrt{2} - 1 = x
\end{aligned}$$

and if $x = -1 + \sqrt{2}$, then

$$\begin{aligned}
\frac{1}{x} - 2 &= \frac{1}{-1+\sqrt{2}} - 2 = \frac{1}{\sqrt{2}-1} - 2 = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} - 2 = \frac{\sqrt{2}+1}{2-1} - 2 \\
&= \frac{\sqrt{2}+1}{1} - 2 = \sqrt{2} + 1 - 2 = \sqrt{2} - 1 = -1 + \sqrt{2} = x
\end{aligned}$$

Both numbers work.

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