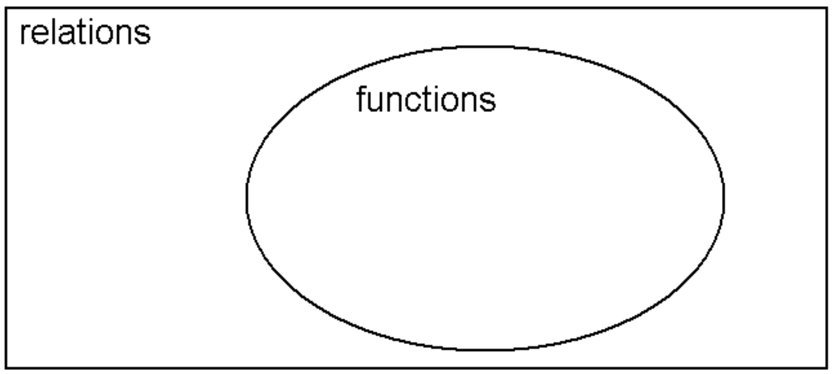
One of the most fundamental and important concepts in mathematics is that of a function. Functions also play a key role in sciences such as physics. What is a function?

Definition: A **relation** is an assignment between elements of a non-empty set called domain and another set called the range. We usually denote relations by lower case letters such as f or g.

Definition: A **function** is an assignment between elements of a non-empty set called domain and another set called the range, with one restriction: that to each element of the domain, only one thing can be assigned. We usually denote functions by lower case letters such as f or g.

We should notice the connection between relations and functions. The definitions are identical, with one exception: the restriction on the assignment in case of functions. So, functions are special relations. Every function is a relation, but not every relation is a function. Another way to say this is that the set of all functions is a subset of the set of all relations.



A relation or function can be given in numerous ways: by a description, by an equation, or a diagram, a table, or a graph. As long as the reader can determine the elements of the domain and range and the assignment, the relation is well defined.

Example 1. Functions given by descrition.

a) Let the domain be the set of all people currently living in the United States. Let us assign to each person their phone number. Is this a function?

The answer is no. This assignment fails the restriction that the assignment must be unique. Since many people have more than one phone number (cell phone, home phone, office, etc.) this assignment is not a function. This is an example of a relation that is not a function.

b) Let the domain be the set of all people currently living in the United States. Let us assign to each person their social security number. Is this a function?

Unless there is some fraud involved, every person has only one social security number. Because of that, this assignment is unique, and so it indeed defines a function.

What is the range in this example? The range is not the set of all 9-digit numbers, only the ones currently assigned to people living in the United States. Once the domain and the assignment are specified, those two determine the range. For this reason, it is usually customary to describe a function by specifying only its domain and assignment.

When we identify a function, we must specify its domain and its assignment.

c) Let the domain be the set of all people currently enrolled in this mathematics class. Let us assign to each person a grade of A or B or C or D or F, based on their current average. If the class has more than 5 students enrolled in it, we are certain that some people will be assigned the same grade. Is this a function?

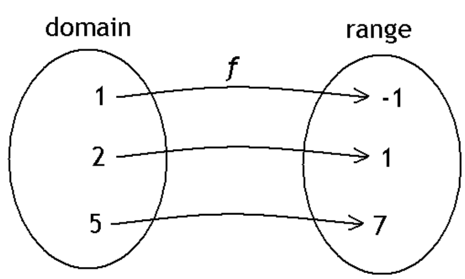
It is a common reaction to think that this is not a function because some people will be assigned the same grade. However, this is a function as long as every student gets only one grade assigned to them. Because the assignment is unique, this is indeed a function. It would still be a function if every student got an A. If this is the case, we say that the function is constant. Constant functions may not be too interesting but they are well defined and even important functions.

d) Another very typical example for a function is that of location as a function of time. Consider an elevator. Suppose we monitor the location of this elevator at every minute for an hour. If the elevator is moving between two floors, we agree to wait until it stops for the first time and record that floor as its location. This is an excellent example for what a function is. It is perfectly normal for the elevator to sit at the same location for several minutes - so it also normal for several time-values in the domain to have the same floor-value assigned. An elevetor sitting on the same floor for an hour would give us a constant function.

What would cause this assignment to fail as a function? The assignment would no longer be a function if there existed a time for which we would have two different floors assigned. For example, at the end of the 5th minute, the elevator is both on the 3rd floor and on the 10th floor. Our physical laws rule out that the same object could be at two different locations at the same time. So, location as a function of time is a very natural application of the concept of functions.

Example 2. Functions given by diagrams.

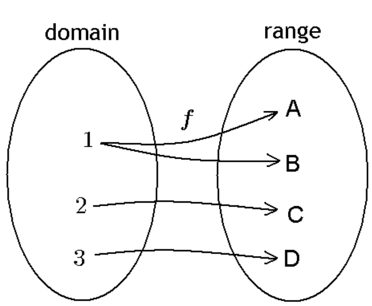
a) The diagram below specifies a function f.

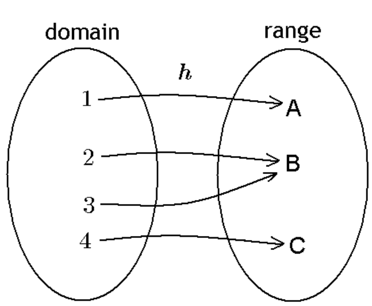


We can see that the domain is a set containing only three elements: $D = \{1, 2, 5\}$ the range is also a three-element set, $R = \{-1, 1, 7\}$ and f assigns -1 to 1, 1 to 2, and 7 to 5. We introduce function notation to describe the assignment with less writing:

$$f(1) = -1, \quad f(2) = 1, \quad \text{and} \quad f(5) = 7$$

b) Let f and h be given as described by the diagrams shown below. Which of them is also a function? Why?





Solution: While h is a function, f is not. f is not a function because it assigns both A and B to 1. On the other hand, h is a function because to each element of the domain, only one thing is being assigned. Note the fact that h is a function even though $h(2) = h(3)$. This kind of "branching" is allowed in case of functions.

Example 3. Functions given as sets of ordered pairs. This involves simply listing all elements of the domain, together with their assigned values.

a) Suppose that f is given as $f = \{(2,5), (3,5), (5,1)\}$. Is this a function?

The set $\{(2,5), (3,5), (5,1)\}$ means that to 2, the value 5 was assigned; to 3, the value 5 was assigned, and to 5, the value 1 was assigned. Using function notation,

$$f(2) = 5, \quad f(3) = 5, \text{ and } f(5) = 1$$

Thus the domain is $\{2,3,5\}$ and the range is $\{1,5\}$. This is indeed a function.

b) Suppose that g is given as $g = \{(1,4), (4,9), (9,1), (1,5)\}$. Is this a function?

g is not a function because $(1,4)$ and $(1,5)$ means $g(1) = 4$ and $g(1) = 5$, so g assigns two values to the number 1.

Example 4. Functions given by tables or equations.

a) A function can also be described using a table. Consider the function f given by the table below.

element of domain	assigned element by f
1	-2
3	6
5	22
-4	13
10	97

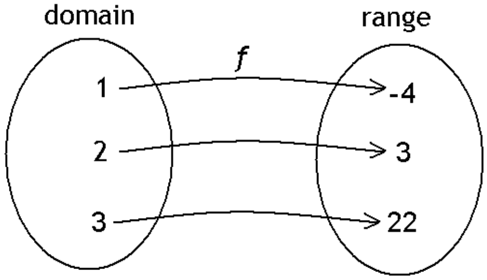
b) Functions can also be described by an equation. Of course, we still have to specify the domain.

Define g as follows: domain: $D = \{-7, -2, 0, 1, 4\}$ and $g(x) = 3x + 1$.

When a function is given in such a way, we can determine the assigned values using the equation of the function. For example, $g(-7)$ is the value assigned to -7. We can compute $g(-7)$ using the definition of g, by replacing x with -7.

$$\begin{aligned} g(x) &= 3x + 1 \\ g(-7) &= 3(-7) + 1 = -20 \qquad \Longrightarrow \qquad g(-7) = -20 \end{aligned}$$

When are two functions equal to each other? Consider now the functions f and g as given below. f is given by a diagram and g is given by an equation.



Define g by domain $\{1, 2, 3, 4\}$ and the assignment $g(a) = a^3 - 5$. We can find the range by computing the values assigned to elements of the domain.

$g(1) = 1^3 - 5 = -4$ $\qquad$ $g(2) = 2^3 - 5 = 3$ $\qquad$ $g(3) = 3^3 - 5 = 22$ $\qquad$ $g(4) = 4^3 - 5 = 59$

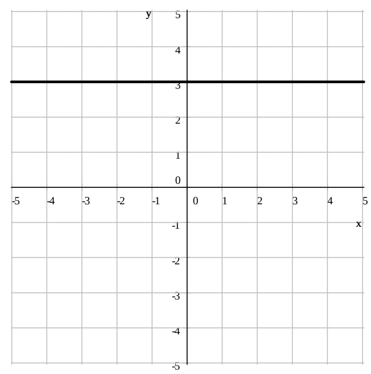
f and g have the same assignment, but have different domains. Can we say that these two functions are the same? Based on the current definition, the answer is no.

Definition: Two functions are equal to each other if they have the same domain and the same assignment.

Basic Functions and Their Properties

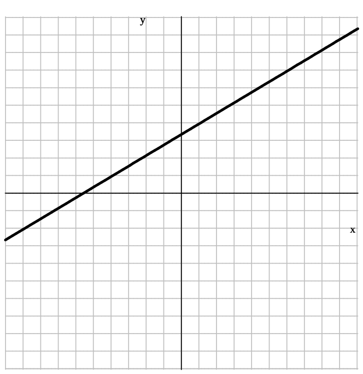
1.) **Constant functions**

$f(x) = c$ where c is a real number.
The graph is a horizontal line.
domain: $\mathbb{R}$ range: $\{c\}$



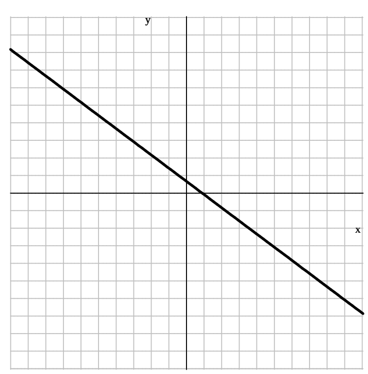
2.) **Linear functions**

$f(x) = mx + b$ where $m \neq 0$. The graph is a straight line.



$m > 0$

domain: $\mathbb{R}$, range: $\mathbb{R}$

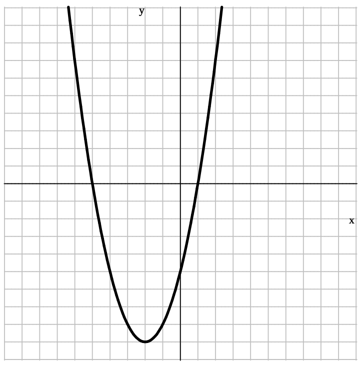


$m < 0$

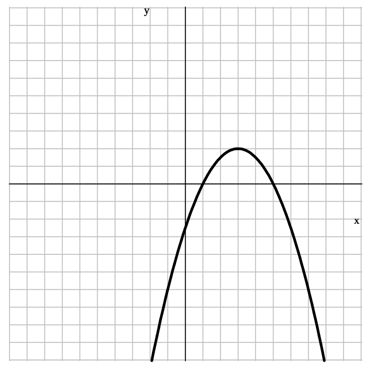
domain: $\mathbb{R}$, range: $\mathbb{R}$

3.) **Quadratic functions** $f(x) = ax^2 + bx + c$ where $a \neq 0$.

The graph is a parabola. It opens upward if $a > 0$ and opens downward if $a < 0$.



$a > 0$



$a < 0$

Case 1. If $a > 0$

Example: $f(x) = x^2 + 4x - 5$

standard form: $f(x) = (x+2)^2 - 9$

factored form: $f(x) = (x+5)(x-1)$

domain: $\mathbb{R}$ range: $[-9, \infty)$

Case 2. If $a < 0$

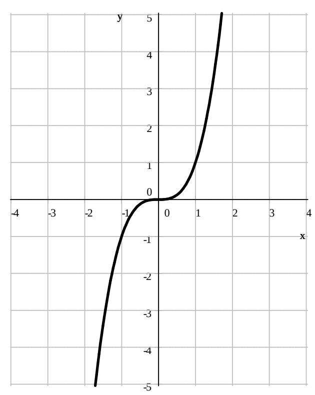
Example: $f(x) = -\frac{1}{2}x^2 + 3x - \frac{5}{2}$

standard form: $f(x) = -\frac{1}{2}(x-3)^2 + 2$

factored form: $f(x) = -\frac{1}{2}(x-1)(x-5)$

domain: $\mathbb{R}$ range: $(-\infty, 2]$

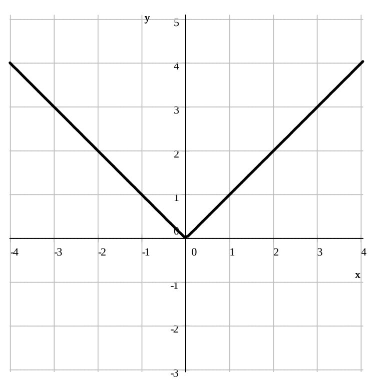
4.) $f(x) = x^3$
domain: $\mathbb{R}$ range: $\mathbb{R}$



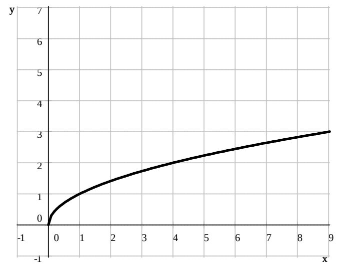
5.) **Absolute value function**

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

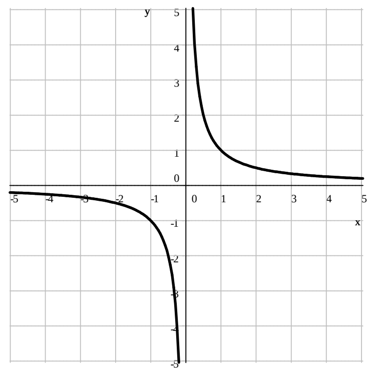
domain: $\mathbb{R}$ range: $[0, \infty)$



6.) $f(x) = \sqrt{x}$
domain: $[0, \infty)$ range: $[0, \infty)$



7.) $f(x) = \dfrac{1}{x}$
domain: (all real numbers except for 0) $\mathbb{R} \setminus \{0\}$
range: (all real numbers except for 0) $\mathbb{R} \setminus \{0\}$



Sample Problems

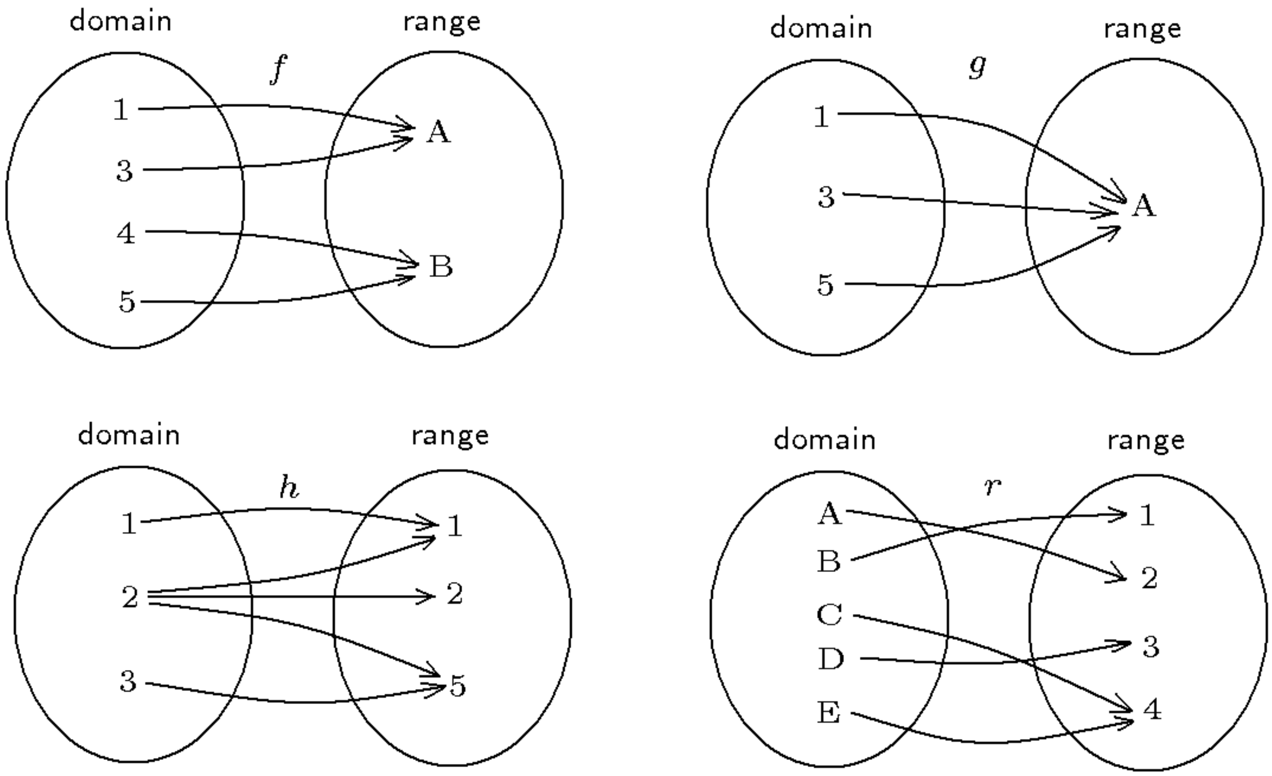
1. Which of the following are functions?

   a) Let the domain be the set of students enrolled in this course. To each student, let us assign the number of minutes they spent studying yesterday for this class.

   b) Let the domain be the set of students enrolled in this course. To each student, let us assign their uncle.

   c) Let the domain be the set $D = \{0, 1, 4, 9\}$. To each number d in the domain, let us assign a number whose square is d. For example, $f(9) = 3$ because $3^2 = 9$.

2. Which of the following are functions?



3. The following relations are given in form of ordered pairs. Which of the following are functions?

   a) $f = \{(1,4), (2,7), (4,1), (5,7), (7,7)\}$

   b) $g = \{(1,1), (-2,4), (2,4), (3,9), (-3,9)\}$

   c) $h = \{(1,1), (3,5), (5,3), (5,5)\}$

4. Define f by its domain being $\mathbb{R}$, the set of all real numbers, and its assignment being $f(x) = -x^2 + 7$. Compute or simplify each of the following expressions.

   a) $f(3)$ b) $f\left(\sqrt{6}\right)$ c) $f(5)$ d) $f(2x)$ e) $f(a-1)$

5. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(t) = t^2 - t + 1$. Compute or simplify each of the following expressions.

   a) $f(0)$ b) $f(-1)$ c) $f(5)$ d) $f(-5)$ e) $f(2t)$

6. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(x) = 2x^2 - 5$. Compute or simplify each of the following expressions.

   a) $f(2)$ b) $f(3)$ c) $f(5)$ d) $f(2) + f(3)$ e) $f(2+3)$

7. In each case, a pair of functions is given. Are they equal or not?

   a) $f = \{(-2,5), (0,1), (2,5), (3,10)\}$ and g is defined by domain $\{-2,0,2,3\}$ and assignment $g(a) = a^2 + 1$.

   b) f is given by the table below

element of domain	assigned element by f
1	0
2	-3
3	-10
5	-24

   and g is defined by domain $\{1,2,3,5\}$ and assignment $g(x) = -x^2 + 1$.

   c) $f = \{(0,-3), (1,2), (2,7), (3,12)\}$ and g is defined by domain $\mathbb{R}$ and assignment $g(m) = 5m - 3$.

Sample Problems - Answers

1. a) function b) not a function c) not a function
2. f, g and r are functions, h is not a function
3. a) function b) function c) not a function
4. a) -2 b) 1 c) -18 d) $-4x^2 + 7$ e) $-a^2 + 2a + 6$
5. a) 1 b) 3 c) 21 d) 31 e) $4t^2 - 2t + 1$
6. a) 3 b) 13 c) 45 d) 16 e) 45
7. a) $f = g$ b) $f \neq g$; $f(3) = -10$ and $g(-3) = -8$ c) $f \neq g$; different domains

Sample Problems - Solutions

1. Which of the following are functions?

   a) Let the domain be the set of students enrolled in this course. To each student, let us assign the number of minutes they spent studying yesterday for this class.

   Solution: Since there is a unique number assigned to each student, this is a function.

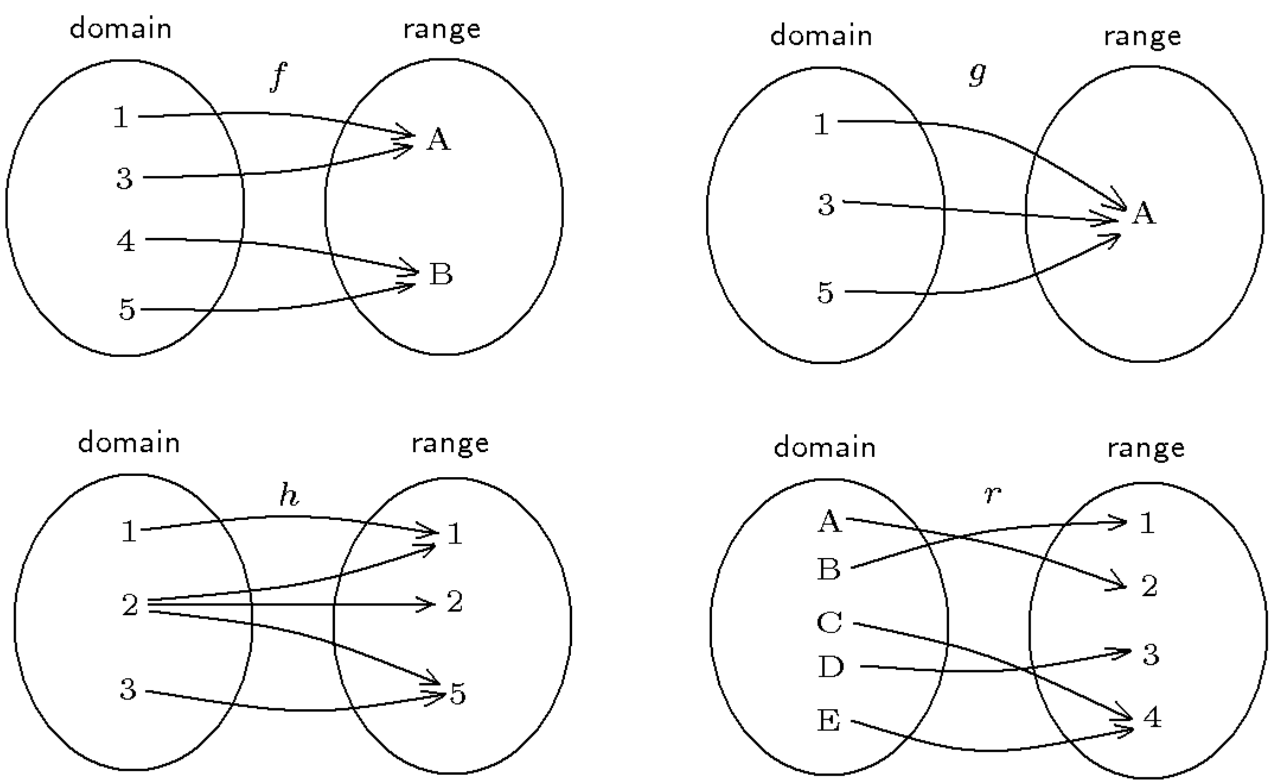
   b) Let the domain be the set of students enrolled in this course. To each student, let us assign their uncle.

   Solution: Most likely this is not a function. If there is just one student in the class who has more than one uncle, then this assignment fails to be a function.

   c) Let the domain be the set $D = \{0, 1, 4, 9\}$. To each number d in the domain, let us assign a number whose square is d. For example, $f(9) = 3$ because $3^2 = 9$.

   Solution: This is not a function because there are more than one values assigned to some elements in the domain. For example, $(9) = 3$ and $f(9) = -3$ because both 3 and -3 results in 9 when squared.

2. Which of the following are functions?



Solution: f is a function because to each element of the domain, only one thing is assigned. Indeed, $f(1) = A$, $f(3) = A$, $f(4) = B$, and $f(5) = B$. The fact that $f(1) = f(3)$ and $f(4) = f(5)$ is OK for a function. This is the 'allowed kind' of 'branching'.

g is also a function because to each element of the domain, only one thing is assigned. g is also called a constant function because all assigned values are the same.

h is not a function because there are several values assigned to 2.

r is also a function because to each element of the domain, only one thing is assigned.

3. The following relations are given in form of ordered pairs. Which of the following are functions?

   a) $f = \{(1,4), (2,7), (4,1), (5,7), (7,7)\}$

   This is a function because to each element of the domain, $D = \{1,2,4,5,7\}$, only one thing is assigned.

   b) $g = \{(1,1), (-2,4), (2,4), (3,9), (-3,9)\}$

   This is a function because to each element of the domain, $D = \{-3,-2,1,2,3\}$, only one thing is assigned.

   c) $h = \{(1,1), (3,5), (5,3), (5,5)\}$

   This is not a function because two numbers are assigned to 5.

4. Define f by its domain being $\mathbb{R}$, the set of all real numbers, and its assignment being $f(x) = -x^2 + 7$. Compute or simplify each of the following expressions.

   a) $f(3) = -2$

   Solution:

   $$f(-3) = -(-3)^2 + 7 = -9 + 7 = -2$$

   b) $f\left(\sqrt{6}\right) = 1$

   Solution:

   $$f\left(\sqrt{6}\right) = -\left(\sqrt{6}\right)^2 + 7 = -6 + 7 = 1$$

   c) $f(5) = -18$

   Solution:

   $$f(5) = -(5)^2 + 7 = -25 + 7 = -18$$

   d) $f(2x) = -4x^2 + 7$

   Solution:

   $$f(2x) = -(2x)^2 + 7 = -4x^2 + 7$$

   e) $f(a-1) = -a^2 + 2a + 6$

   Solution:

   $$f(a-1) = -(a-1)^2 + 7 = -\left(a^2 - 2a + 1\right) + 7 = -a^2 + 2a - 1 + 7 = -a^2 + 2a + 6$$

5. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(t) = t^2 - t + 1$. Compute or simplify each of the following expressions.

   a) $f(0) = 1$

   Solution: $f(0) = 0^2 - 0 + 1 = 1$

   b) $f(-1) = 3$

   Solution: $f(-1) = (-1)^2 - (-1) + 1 = 3$

   c) $f(5) = 21$

   Solution: $f(5) = (5)^2 - (5) + 1 = 21$

   d) $f(-5) = 31$

   Solution: $f(-5) = (-5)^2 - (-5) + 1 = 31$

   e) $f(2t) = 4t^2 - 2t + 1$

   Solution: $f(2t) = (2t)^2 - (2t) + 1 = 4t^2 - 2t + 1$

6. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(x) = 2x^2 - 5$. Compute or simplify each of the following expressions.

   a) $f(2) = 3$

   Solution: $f(2) = 2 \cdot 2^2 - 5 = 2 \cdot 4 - 5 = 8 - 5 = 3$

   b) $f(3) = 13$

   Solution: $f(3) = 2 \cdot 3^2 - 5 = 2 \cdot 9 - 5 = 18 - 5 = 13$

   c) $f(5) = 45$

   Solution: $f(5) = 2 \cdot 5^2 - 5 = 2 \cdot 25 - 5 = 50 - 5 = 45$

   d) $f(2) + f(3) = 16$

   Solution: We have just computed the values of $f(2)$ and $f(3)$. $f(2) + f(3) = 3 + 13 = 16$

   e) $f(2+3) = 45$

   Solution: $f(2+3) = f(5)$ which we have already computed and found to be 45.

   This problem illustrates the fact that most of the time $f(a) + f(b)$ and $f(a+b)$ are different.

7. In each case, a pair of functions is given. Are they equal or not?

   a) $f = \{(-2,5), (0,1), (2,5), (3,10)\}$ and g is defined by domain $\{-2,0,2,3\}$ and assignment $g(a) = a^2 + 1$.

   Solution: these functions are the same because they have the same domain and the same assignment. So, $f = g$.

   b) f is given by the table below

element of domain	assigned element by f
1	0
2	-3
3	-10
5	-24

   and g is defined by domain $\{1,2,3,5\}$ and assignment $g(x) = -x^2 + 1$.

   Solution: these functions are the not same because they do not have the same assignment: $f(3) = -10$ and $g(-3) = -8$. So, $f \neq g$.

   c) $f = \{(0,-3), (1,2), (2,7), (3,12)\}$ and g is defined by domain $\mathbb{R}$ and assignment $g(m) = 5m - 3$.

   Solution: these functions are the not same because they do not have the same domain. The domain of f is quite small, it contains only four numbers. Meanwhile, the domain of g is $\mathbb{R}$,the set of all real numbers. So, $f \neq g$.

