Part 1- What is Mathematics?

As a graduate student, I had the annoying habit of asking my teachers and peer students what they think mathematics is. To my surprise, I received many different answers, and, to this day, I agree with many of them. In my eyes, mathematics is many things. In mathematics, we will be talking a lot about things being true or being not true. Although this probably happens in every course in every discipline, mathematical truth can be objectively established and agreed upon. To achieve such an objective approach, we have to develop a language that is objectively understood. In this sense, mathematics is also a language.

Definition: **Mathematics** is a collection of *true statements* that are developed, expressed, and interpreted using an objective *language* and rules of *logic*.

To understand mathematics, we need to first agree on an objective language. Reading and writing mathematical notation correctly will be important.

'Then you should say what you mean', the March Hare went on. 'I do,' Alice hastily replied; 'at least - at least I mean what I say - that's the same thing, you know.' 'Not the same thing a bit!' said the Hatter. Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see!'

Lewis Carroll

Alice's Adventures in Wonderland

Part 2 - The Language of Mathematics

Mathematical statements are much like English sentences. As English sentences are built from different kinds of words, mathematical statements usually contain three types of components: **objects, operations, and relations**.

Definition: The concept of an **object** (very much like nouns in English sentences) is usually clearly understood and needs no explanation.

Examples of objects from algebra include the number 2, the number 3, and numbers in general. Objects from geometry include lines, points, triangles, circles, line segments, etc.

Definition: An **operation** is an action (very much like verbs in English sentences) that can be applied to objects and usually result in new objects.

Examples of operations from algebra include addition, subtraction, multiplication, and division. Operations from geometry include reflection to a line, rotation, or translation that can be performed on points, triangles, circles, line segments, etc.

Definition: A **relation** is something we use to compare two objects.

Unlike operations, relations do not produce new things; we use relations to compare already existing objects. For example, the operation addition produces 7 if applied to 2 and 5. Relations in algebra are equal, or less, or greater. Relations from geometry are how geometric objects can be compared to each other: similar, congruent, parallel, perpendicular.

Many statements use at least one of each of these three components. In the statement $2+5=7$ the numbers 2, 5, and 7 are the objects, addition (denoted by $+$) is the operation, and being equal (denoted by $=$) is the relation.

It is a common misconception to think of mathematics as the study of *only* numbers. Numbers are only certain types of objects. As we progress in the study of mathematics, we will find that there are many other types of interesting objects. For example, sets are objects we will soon study. Furthermore, the study of operations and relations is also interesting and fruitful.

So, what kind of true statements can be established in mathematics? There are three types of true statements in mathematics: **definitions**, **axioms**, and **theorems**.

Definition: A **definition** is a labeling statement in which we agree to use an expression to refer to an object, operation, or relation in mathematics.

Definitions are all true statements, because they simply reflect an agreement in the terms of the language. To be precise, these decisions were made without consulting any of us, often decades (if not centuries) before any of us were ever born. An example of a definition would be if we pointed to a clear sky and said: "From now on, let's call this color blue".

Definition: A **theorem** is a statement that we insist on proving before believing that it is true. To **prove** a theorem means to derive it from previously established true statements, using logically correct steps.

If you think about that last definition a little, you will see that no theorem can exist, unless we agree on accepting a few statements to be true, without proving them. These are our "starting true statements".

Definition: An **axiom** is a statement we agree to accept to be true without proof.

Axioms are usually simple, basic statements that are in agreement with our intuition. For example, the statement "*It is possible to draw a straight line from any point to any other point.*" is an axiom. It has been a constant effort to keep the number of axioms to a minimum. We prove a theorem by deriving its statement from statements already established to be true.

To be precise, when we prove our first theorem, we derive its statement from the axioms. When we prove our second theorem, we derive its statement from the axioms and the first theorem. When proving the third theorem, we can use all the axioms, and the first and second theorems. And so on. For our tenth theorem, we have all the axioms and the first nine theorems at our disposal. At this point, we are building a logically sound theory, a unified discipline within mathematics. It is one thing to suspect, to feel, or to have a hunch that something is true. It is entirely different from proving it, with unescapable force of logic.

The ancient Greek mathematician Euclid discussed mathematics in this manner, i.e stating axioms and building a theory by deriving a sequence of theorems from the axioms. (He called axioms postulates.) Mathematicians immediately accepted and embraced this logical approach to the study of mathematics - and it is how it is done still today. Although Euclid has contributed to several parts of mathematics (including geometry and number theory), he completely axiomatized of what we now call classical geometry or Euclidean geometry. He stated five postulates, accepted them to be true and derived most basic theorems of classical geometry.

Part 3 - The Real Number System

We will be studying the properties of the real numbers. Most statements about the real numbers are theorems, and so they can be derived from just a few axioms. These axioms are behind many properties, and we will refer to them often as we review beginning algebra. The axioms of the real numbers are as follows.

A1. Addition is commutative.
For all real numbers x and y,
$$x+y=y+x$$

A2. Addition is associative.
For all real numbers x,y, and z,
$$(x+y)+z=x+(y+z).$$

A3. Additive identity.
There exists a real number d such that for all real numbers x,
$$x+d=x.$$
(This number is 0).

A4. Additive inverse.
For all real numbers x, there exists a real number x^*, such that
$$x+x^*=0.$$
(We denote this number by $-x$).

M1. Multiplication is commutative.
For all real numbers x and y,
$$xy=yx.$$

M2. Multiplication is associative.
For all real numbers x,y, and z,
$$(xy)z=x(yz).$$

M3. Multiplicative identity.
There exists a real number d such that for all real numbers x,
$$xd=x.$$
(This number is 1.)

M4. Multiplicative inverse.
For all non-zero real number x, there exists a real number x^* such that
$$xx^*=1.$$
(We denote this number by $\frac{1}{x}$).

D1. The Distributive Law.
For all real numbers x,y, and z,
$$z(x+y)=zx+zy.$$

Sets that have these properties are called fields and are studied in depth in abstract algebra. We will often re-phrase axiom A4 as '**To subtract is to add the opposite.**' and M4 as '**To divide is to multiply by the reciprocal.**'. We will often re-phrase A3 and M3 as '**Two operations that we can perform on any number without changing its value are: to add zero and to multiply by one.**'

There are also some additional axioms of the real number system about ordering. They are listed here for interested students, but we will mostly focus on the nine axioms listed before.

O1. For all real numbers a and b, either $a \leq b$ or $b \leq a$ or both.

O2. For all real numbers a and b, If $a \leq b$ and $b \leq a$, then $a = b$.

O3. For all real numbers a, b, and c, if $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. For all real numbers a, b, and c, if $a \leq b$ then $a + c \leq b + c$.

O5. For all real numbers a, b, and non-negative c, if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

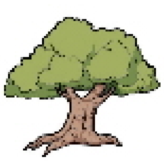
A set with all these properties is called an ordered field. However, so far these properties hold for both $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers.) What distinguishes these two is the completeness property, that is true for $\mathbb{R}$ but not for $\mathbb{Q}$. This property is studied in higher level mathematics classes such as calculus.

C. Completeness property: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Identity and Inverse elements

An **identity element** does 'nothing'. It is a unique elemet of the set that works for every element. The **inverse of an element** is another element such that when the operation is applied to the number and its inverse, the result is the identity. The inverse is different for different numbers.

		Real Numbers with Addition.	Real Numbers with Multiplication
Identity	Systematic Name	additive identity	multiplicative identity
	Defining Property	does 'nothing' in addition	does 'nothing' in multiplication
	Value	0	1
Inverse	Systematic Name	additive inverse of a	multiplicative inverse of a
	Non-Systematic Name	opposite of a	reciprocal of a
	Defining Property	$a + (-a) = 0$ i.e. 'takes' a to the identity	$a \cdot \frac{1}{a} = 1$ i.e. 'takes' a to the identity



Enrichment

1. Look up Euclid's Elements on the internet. (Start at Wikipedia). What is Elements? What is Euclid's contribution to **all** subjects within today's mathematics? What are postulates? List Euclid's five postulates. Explain the significance of these five statements.

2. What is the parallel postulate? Look up the history of Euclid's parallel postulate on the internet. (Start with Wikipedia.) How would we go about proving that an axiom is really a theorem? List statements from geometry that are logically equivalent to the parallel postulate. What exactly could it mean for two axioms to be equivalent to each other?

~325 B.C. – 270 B.C

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.