Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5

e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

Example 2. List all positive factors of the number 28.

Solution: We start counting, starting at 1.
Is 1 a divisor of 28? Yes, because $28 = 1 \cdot 28$.
We note both factors we found.

28

1	28

We continue counting. Is 2 a divisor of 28?
Yes, because $28 = 2 \cdot 14$.
We note both factors we found.

28

1	28
2	14

We continue counting. Is 3 a divisor of 28? No.
We can divide 28 by 3 and the answer is not an integer.

We continue counting. Is 4 a divisor of 28? Yes, because $28 = 4 \cdot 7$. We note both factors we found.

28

1	28
2	14
4	7

We continue counting. Is 5 a divisor of 28? No. (We can check with the calculator.) Is 6 a divisor of 28? No. Now we arrive to 7, a number that is already listed as a factor. That's our signal that we have found all of the divisors of 28. We list the divisors in order:

factors of 28: $\boxed{1, 2, 4, 7, 14, 28}$

Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7 so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number. Prime numbers are a fascinating study within mathematics. Let us first recall the definition. Given a number n, we can find all of its divisors. For example, $n = 20$ has six divisors: $1, 2, 4, 5, 10$, and 20. Prime numbers have a very short list of divisors: only the trivial divisors, 1 and the number itself. For example, 7 is a prime number. 6 is not a prime number since it has divisors other than 1 and 6.

It is important to notice that 1 is not a prime number. The first few prime numbers are: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$ With respect to multiplication, prime numbers are the basic building blocks of numbers.

Theorem: (*Fundamental Theorem of Arithmetic*) Every integer greater than 1 can be written as a product of prime numbers, and this decomposition is unique up to order of factors.

For example, $20 = 2 \cdot 2 \cdot 5$. According to the fundamental theorem of arithmetic, there is no other way to write 20 as a product of prime numbers. We usually use exponential notation: $20 = 2^2 \cdot 5$.

Example 3. Find the prime factorization of 180.

Solution: We start with the smallest prime number, 2 and ask: is 180 divisible by 2? If the answer is yes, we write down and divide 180 by 2.

$$\begin{array}{l|l} 180 & 2 \\ 90 & \end{array}$$

Now we ask: is 90 divisile by 2? The answer is yes. So we write 2 next to 90 and divide by 2.

$$\begin{array}{l|l} 180 & 2 \\ 90 & 2 \\ 45 & \end{array}$$

Now we ask: is 45 divisile by 2? The answer is no. So, we are done with the prime factor 2 and move on to the next prime number, 3. Since 45 is divisible by 3, we write it next to 45 and divide by 3.

$$\begin{array}{l|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & \end{array}$$

Now we ask: is 15 divisile by 3? The answer is yes. So we write 3 next to 15 and divide by 3.

$$\begin{array}{l|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & \end{array}$$

The only prime divisor of 5 is 5 itself. We write 5 next to 5 and then divide.

$$\begin{array}{l|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

Once we wrote 1, we are done. The prime factorization of 180 is therefore

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \text{ or } 180 = 2^2 \cdot 3^2 \cdot 5$$

Example 4. Find the prime factorization of 300.

We start with the first prime number, 2. Is our number, 300 divisible by 2? If yes, we divide 300 by 2. Since $300 = 2 \cdot 150$, we now have one prime factor, 2 and we must find the prime factorization of 150.

$$\begin{array}{r|l} 300 & 2 \\ 150 & 2 \\ 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array}$$

We ask next: is the number 150 divisible by 2? If yes, we divide 150 by 2. So now $300 = 2 \cdot 2 \cdot 75$ and we are looking for the prime factorization of 75.

We ask next: is the number 75 divisible by 2? This time, the answer is no. We have exhausted the prime factor 2. So we roll up to 3 and ask: is 75 divisible by 3? If yes, we divide 75 by 3. Since $75 \div 3 = 25$, so now $300 = 2 \cdot 2 \cdot 3 \cdot 25$ and we are looking for the prime factorization of 25. Although we know the final answer now, we continue the process. Every time we find a factor, we write it down and divide. The quotient is written down under the pair in the first column. Once that column reaches 1, the second column is the prime factorization of our number. Thus $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = \boxed{2^2 \cdot 3 \cdot 5^2}$

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem: A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.
A number n is divisible by 5 if its last digit is 0, or 5.
A number n is divisible by 4 if the two-digit number formed of its last two digits is divisible by 4.
A number n is divisible by 3 if the sum of its all digits is divisible by 3.
A number n is divisible by 9 if the sum of its all digits is divisible by 9.

Practice Problems

1. List all the factors of 48.
2. Which of the following is NOT a prime number?

   $53, \ 73, \ 91, \ 101, \ 139$

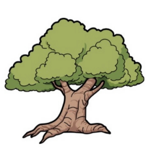
3. Consider the following numbers.

   $64, \ 75, \ 80, \ 128, \ 270$

   a) Find all numbers on the list that are divisible by 5.
   b) Find all numbers on the list that are divisible by 3.
   c) Find all numbers on the list that are divisible by 4.

4. Find the prime factorization for each of the following numbers.

   a) 600 b) 5500 c) 2016 d) 2015



Enrichment

1. Suppose that given a number n, we need to determine whether it is a prime number or not. Until what number must we check all the prime numbers whether they are a divisor of n or not? When can we stop and say that this number must be a prime?
2. Magic: think of a three digit number. Enter a six-digit number into your calculator by repeating your three-digit number twice. For example, if you thought of the three-digit number 275, then enter 275275 into your calculator.

   Done? No matter what number you used to start, the number in your calculator is divisible by 7. Divide by 7. The number in your calculator now is still divisible by 11. Divide it by 11. The number in your calculator is still divisible by 13. Divide it by 13. What do you see? Can you explain it?

3. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
4. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

Answers for Practice Problems

1. $1, 2, 3, 4, 6, 8, 12, 16, 24, 48$ 2. 91 3. a) $80, 75, 270$ b) $75, 270$ c) $128, 80, 64$

4. a) $600 = 2^3 \cdot 3 \cdot 5^2$ b) $5500 = 2^2 \cdot 5^3 \cdot 11$ c) $2016 = 2^5 \cdot 3^2 \cdot 7$ d) $2015 = 5 \cdot 13 \cdot 31$