Example 1: Graph the parabola $y = x^2 - 8x + 7$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We start with algebra. We obtain all three form of the equation first. These three forms are the polynomial form, the standard form, and the factored form. The polynomial form was given.

$$y = x^2 - 8x + 7 \implies \textbf{polynomial form}$$

We then complete the square. Half of the linear coefficient is -4, thus we work out $(x-4)^2$.

$$(x-4)^2 = (x-4)(x-4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

We are now ready to complete the square:

$$\begin{aligned} y &= x^2 - 8x + 7 & (x-4)^2 = x^2 - 8x + 16 \\ y &= \underbrace{x^2 - 8x + 16} - 16 + 7 & \\ y &= (x-4)^2 - 9 \implies \textbf{standard form} & \end{aligned}$$

We factor via the difference of squares theorem

$$\begin{aligned} y &= (x-4)^2 - 3^2 \quad \text{since } 3^2 = 9 \\ y &= (x-4+3)(x-4-3) \\ y &= (x-1)(x-7) \implies \textbf{factored form} \end{aligned}$$

From the polynomial form, we easily obtain the $y-$intercept, $(0,7)$, since

$$\text{If } x = 0, \text{ then } y = 0^2 - 8(0) + 7 = 7 \implies \text{found } (0,7)$$

From the standard form, we obtain the coordinates of the vertex.

$$y = (x-4)^2 - 9$$

The trick is to think of the vertex as the point on the graph where the quadratic expression achieves its lowest possible value. Let us start with $(x-4)^2$. Because it is a square, $(x-4)^2$ will be non-negative for every value of x. The lowest possible value of $(x-4)^2$ is zero, when we square zero. This means that $x - 4 = 0$ must hold, thus $x = 4$.
In short, the lowest value of $(x-4)^2$ is zero, when $x = 4$.
Consequently, the lowest possible value of $(x-4)^2 - 9$ is -9, when $x = 4$. Thus the coordinates of the vertex are $(4, -9)$.
The factored form tells us the coordinates of the $x-$intercepts. For the $x-$intercepts, we have to solve the equation

$$\begin{aligned} x &= ? \quad \text{so that } y = 0 \\ x &= ? \quad \text{so that } x^2 - 8x + 7 = 0 \end{aligned}$$

$$\begin{aligned} 0 &= x^2 - 8x + 7 \\ 0 &= (x-1)(x-7) \implies x_1 = 1 \quad x_2 = 7 \end{aligned}$$

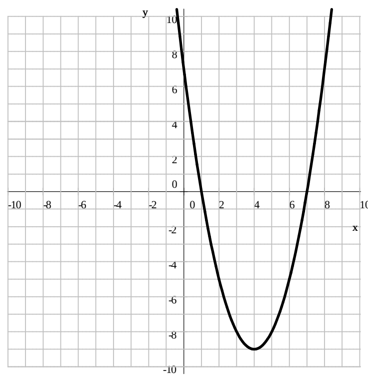
Thus $y = 0$ when $x = 1$ and $x = 7$. Thus, there are two $x-$intercepts, $(1,0)$ and $(7,0)$.
We will compute a few more points before graphing the parabola. To avoid finding points with too large coordinates, we will work with $x-$coordinates, close to the vertex. The $y-$coordinate can be found by substituting

values for x into any of the three forms of the equations to find y. This time we will work with the polynomial form.

$$\begin{aligned}
\text{if } x &= 2, \text{ then } y = (2)^2 - 8(2) + 7 = 4 - 16 + 7 = -5 \quad \Longrightarrow \quad \text{found} \quad (2,-5)\\
\text{if } x &= 3, \text{ then } y = (3)^2 - 8(3) + 7 = 9 - 24 + 7 = -8 \quad \Longrightarrow \quad \text{found} \quad (3,-8)\\
\text{if } x &= 4, \text{ then } y = -9 \quad \text{already found} \quad (4,-9)\\
\text{if } x &= 5, \text{ then } y = (5)^2 - 8(5) + 7 = 25 - 40 + 7 = -8 \quad \Longrightarrow \quad \text{found} \quad (5,-8)\\
\text{if } x &= 6, \text{ then } y = (6)^2 - 8(6) + 7 = 36 - 48 + 7 = -5 \quad \Longrightarrow \quad \text{found} \quad (6,-5)
\end{aligned}$$

We are ready to graph: we have the following points, listed left to right:

$y-$intercept	$(0,7)$
$x-$intercept	$(1,0)$
	$(2,-5)$
	$(3,-8)$
vertex	$(4,-9)$
	$(5,-8)$
	$(6,-5)$
$x-$intercept	$(7,0)$



Example 2: Graph the parabola $y = x^2 + 6x + 5$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution:.

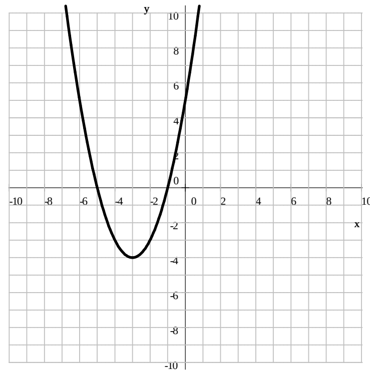
$$\begin{aligned}
y &= x^2 + 6x + 5 \quad \Longrightarrow \quad \textbf{polynomial form} \quad \Longrightarrow \quad y - \text{intercept: } (0,5)\\
y &= x^2 + 6x + 5 \qquad\qquad (x+3)^2 = x^2 + 6x + 9\\
y &= \underbrace{x^2 + 6x + 9} - 9 + 5\\
y &= (x+3)^2 - 4 \quad \Longrightarrow \quad \textbf{standard form} \quad \Longrightarrow \quad \text{vertex: } (-3,-4)\\
y &= (x+3)^2 - 2^2\\
y &= (x+3+2)(x+3-2)\\
y &= (x+5)(x+1) \quad \Longrightarrow \quad \textbf{factored form} \quad \Longrightarrow \quad x - \text{intercepts } (-5,0), \ (-1,0)
\end{aligned}$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the standard square form of the equation, $y = (x+3)^2 - 4$.

$$\begin{aligned}
\text{if } x &= -5, \text{ then } y = 0 \quad \text{already found} \quad (-5,0)\\
\text{if } x &= -4, \text{ then } y = (-4+3)^2 - 4 = (-1)^2 - 4 = 1 - 4 = -3 \quad \Longrightarrow \quad \text{found} \quad (-4,-3)\\
\text{if } x &= -3, \text{ then } y = -4 \quad \text{already found} \quad (-3,-4)\\
\text{if } x &= -2, \text{ then } y = (-2+3)^2 - 4 = 1^2 - 4 = 1 - 4 = -3 \quad \Longrightarrow \quad \text{found} \quad (-2,-3)\\
\text{if } x &= -1, \text{ then } y = 0 \quad \text{already found} \quad (-1,0)
\end{aligned}$$

We are ready to graph: we have the following points, listed left to right:

$x-$intercept	$(-5,0)$
	$(-4,-3)$
vertex	$(-3,-4)$
	$(-2,-3)$
$x-$intercept	$(-1,0)$
$y-$intercept	$(0,5)$



Example 3: Graph the parabola $y = x^2 - 2x - 15$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

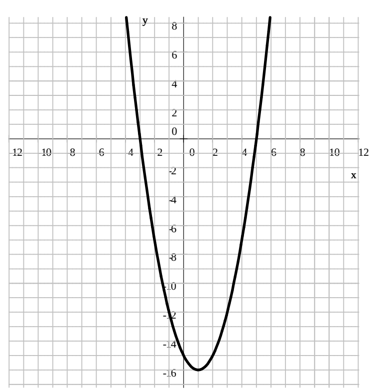
Solution:.

$$\begin{aligned}
y &= x^2 - 2x - 15 && \Longrightarrow \quad y - \text{intercept: } (0,-15) \\
y &= x^2 - 2x - 15 && \qquad (x-1)^2 = x^2 - 2x + 1 \\
y &= \underbrace{x^2 - 2x + 1} - 1 - 15 \\
y &= (x-1)^2 - 16 && \Longrightarrow \quad \text{vertex: } (1,-16) \\
y &= (x-1)^2 - 4^2 \\
y &= (x-1+4)(x-1-4) \\
y &= (x+3)(x-5) && \Longrightarrow \quad x - \text{intercepts } (-3,0),\ (5,0)
\end{aligned}$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the factored form of the equation, $y = (x+3)(x-5)$.

$$\begin{aligned}
& (-3,0) \quad \text{already found} \\
\text{if } x &= -1, \text{ then } y = (-1+3)(-1-5) = 2(-6) = -12 \quad \Longrightarrow \quad (-1,-12) \\
& (0,-15) \quad \text{already found} \\
& (1,-16) \quad \text{already found} \\
\text{if } x &= 2, \text{ then } y = (2+3)(2-5) = 5(-3) = -15 \quad \Longrightarrow \quad (2,-15) \\
\text{if } x &= 3, \text{ then } y = (3+3)(3-5) = 6(-2) = -12 \quad \Longrightarrow \quad (3,-12) \\
& (5,0) \quad \text{already found}
\end{aligned}$$

additional points can be similary found: $(-4,9)$ and $(6,9)$.



Practice Problems

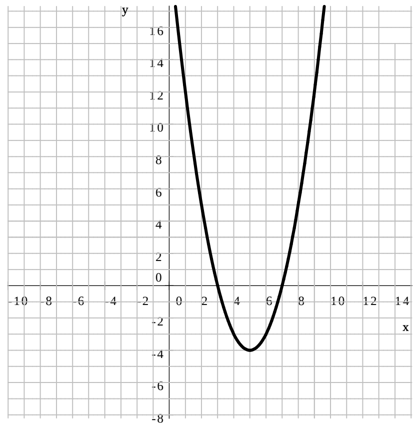
Graph each of the parabolas given below. In each case, clearly label the coordinates of five points of the parabola, including vertex and intercepts.

1. $y = x^2 - 10x + 21$
2. $y = x^2 + 4x - 5$
3. $y = x^2 - 4x + 3$

Practice Problems - Answers

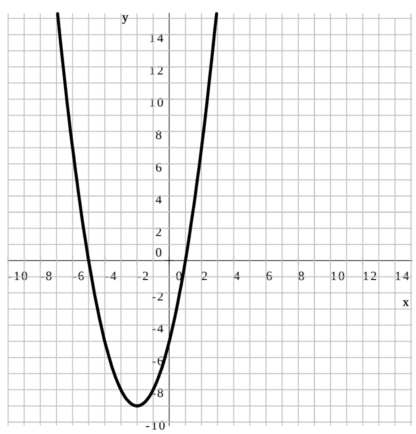
1. $y = x^2 - 10x + 21$

$$\begin{aligned} y &= x^2 - 10x + 21 & y-\text{intercept: } (0,21) \\ y &= (x-5)^2 - 4 & \text{vertex: } (5,-4) \\ y &= (x-3)(x-7) & x-\text{intercepts: } (3,0) \text{ and } (7,0) \end{aligned}$$



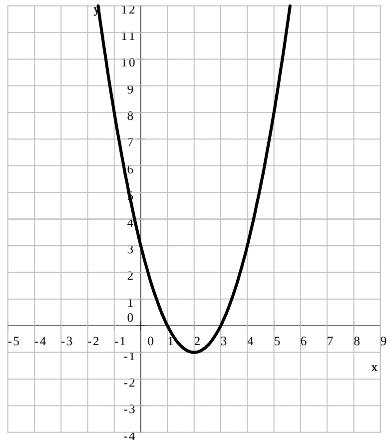
2. $y = x^2 + 4x - 5$

$$\begin{aligned} y &= x^2 + 4x - 5 & y-\text{intercept: } (0,-5) \\ y &= (x+2)^2 - 9 & \text{vertex: } (-2,-9) \\ y &= (x+5)(x-1) & x-\text{intercepts: } (-5,0) \text{ and } (1,0) \end{aligned}$$



3. $y = x^2 - 4x + 3$

$$\begin{aligned} y &= x^2 - 4x + 3 && y-\text{intercept: } (0,3) \\ y &= (x-2)^2 - 1 && \text{vertex: } (2,-1) \\ y &= (x-1)(x-3) && x-\text{intercepts: } (1,0) \text{ and } (3,0) \end{aligned}$$



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