Linear expression such as $2x+1$ have no smallest or greatest values. Substituting greater and greater positive numbers (as x) into $2x+1$ produces greater and greater values. Similarly, substituting large negative numbers (as x) into $2x+1$ produces negative values with increasing absolute value.

Quadratic expressions are fundamentally different, as no square of any real number is negative. Consequently, most quadratic expressions have a smallest possible value: zero.

Example 1. What is the smallest value of each of the following quadratic expressions?

a) x^2 b) $(x-4)^2$ c) $(2x+1)^2$ d) $(x+7)^2$

Solution: a) No real number can have a negative square. Consequently, if x^2 could take the value zero, that would be the smallest value. That is possible, when we square $x=0$. Thus, the lowest value of x^2 is zero, when x is zero.

b) Recall that $(x-4)^2$ is a complete square; a difference squared. No matter what the value of x is, $x-4$ is a real number and so it can NOT have a negative square. Consequently, if $(x-4)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x-4)^2$ is zero, when $x-4$ is zero, that is, when $x=4$. (Just solve the linear equation $x-4=0$). In short: the lowest value of $(x-4)^2$ is zero, when $x=4$.

c) Recall that $(2x+1)^2$ is a square, thus no value for x will ever result in a negative value. Consequently, if $(2x+1)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(2x+1)^2$ is zero, when $2x+1=0$. We solve the linear equation $2x+1=0$ for x and obtain $x=-\frac{1}{2}$. So, the lowest value of $(2x+1)^2$ is zero, when $x=-\frac{1}{2}$.

d) Since $(x+7)^2$ is a square, no value for x will ever result in a negative value. Consequently, if $(x+7)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x+7)^2$ is zero, when $x+7=0$. We solve the linear equation $x+7=0$ for x and obtain $x=-7$. So, the lowest value of $(x+7)^2$ is zero, when $x=-7$.

Example 2. Find the smallest value of each of the following quadratic expressions.

a) x^2+25 b) $(x-4)^2-1$ c) $(2x+1)^2+12$ d) $(x+7)^2-100$

Solution: First imagine a room where some people gathered. Everyone empties their vallets and pockets and count all their cash and then compare. As it turns out, Mr. X has the least amount of money on them, only \$1.50. Then another hour later, everyone in the room receives \$20. Who has the least amount of money now? The answer is clearly Mr. X, this time with \$21.50.

a) Consider the expression x^2+25. Recall that the lowest value of x^2 is zero, when x is zero. Then the lowest value of x^2+25 is 25, when x is zero. (Imagine that the 'poorest' person in the room was Ms. Y, with no money on her. If then everyone in the room receives \$25, she would still be the one with the smallest amount of money, exactly \$20.)

b) Consider now the expression $(x-4)^2-1$. The lowest value of $(x-4)^2$ is zero, when $x=4$. Then the lowest value of $(x-4)^2-1$ is -1, when $x=4$.

c) Consider now the expression $(2x+1)^2+12$. Because it is a square, the lowest value of $(2x+1)^2$ is zero, when $x=-\dfrac{1}{2}$. Then the lowest value of $(2x+1)^2+12$ is 12, when $x=-\dfrac{1}{2}$.

d) Consider now the expression $(x+7)^2-100$. Because it is a square, the lowest value of $(x+7)^2$ is zero, when $x=-7$. Then the lowest value of $(x+1)^2-100$ is -100, when $x=-7$.

Example 3. Find the smallest value of each of the following quadratic expressions.

a) x^2+2x-5 b) $x^2-8x+15$ c) x^2-12x d) x^2-100

Solution: a) We will apply the same ideas, but first we need to transform this expression into a more suitable form. For that, We simply complete the square. Half of the linear coefficient is 1, so our suitable complete square is $(x+1)^2$.

$$\begin{aligned} E &= x^2+2x-5 && \text{helper line: } (x+1)^2 = x^2+2x+1 \\ &= \underbrace{x^2+2x+1}-1-5 && \text{so we smuggle in } 1 \\ &= (x+1)^2-6 \end{aligned}$$

This form of a quadratic expression is often called **the standard form**. Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x+1)^2-6$ is -6, when $x=-1$.

b) Consider the expression $x^2-8x+15$. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2-8x+15 && \text{helper line: } (x-4)^2 = x^2-8x+16 \\ &= \underbrace{x^2-8x+16}-16+15 && \text{so we smuggle in } 16 \\ &= (x-4)^2-1 \end{aligned}$$

Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x-4)^2-1$ is -1, when $x=4$.

c) Consider the expression x^2-12x. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2-12x && \text{helper line: } (x-6)^2 = x^2-12x+36 \\ &= \underbrace{x^2-12x+36}-36 && \text{so we smuggle in } 36 \\ &= (x-6)^2-36 \end{aligned}$$

The smallest value of x^2-12x is -36, when $x=6$.

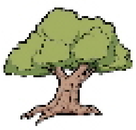
d) Consider the expression x^2-100. We do not complete the square as the expression is already in its standard form. If it helps, we can think of x^2-100 as $(x-0)^2-100$. Either way, the smallest value of x^2-100 is -100, when $x=0$.

Completing the square is not just a factoring technique. It is really our only way (at this point) to determine the smallest value that the expression takes. Completing the square is a way to understand quadratic expressions.

Practice Problems

Find the smallest value of each of the following expressions.

1. $x^2 - 4x + 7$
2. $x^2 + 10x + 14$
3. $x^2 - 2x$
4. $x^2 + 6x + 42$
5. $x^2 - 100x + 60$
6. $x^2 + 8x + 20$
7. $x^2 - 16$
8. $x^2 - 16x + 4$
9. $x^2 - 4x + 4$
10. $x^2 + 18x + 81$

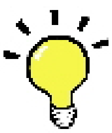


Enrichment

1. Not all squares produce zero as their smallest value. Consider the expression $\left(x^2+3\right)^2$. What is the smallest value of this expression? How about $\left(x^2+3\right)^2 - 1$? And $x^4 + 10x^2 + 14$?

2. Until now, the quadratic expressions we saw all had a leading coefficient of 1. How is our argument modified with different leading coefficients? Discuss the smallest value of each of the given expressions.

   a) $3(x-5)^2 + 8$ b) $-2(x+1)^2 - 49$ c) $5\left((x+4)^2 - 25\right)$



Answers

Practice Problems

1\. 3 2. -11 3. -1 4. 33 5. -40 6. 4 7. -16 8. -60 9. 0 10. 0

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