Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

Definition: Let N be a non-negative number. Then the **square root of** N (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N. If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

Square roots, when stretched over entire expressions, also serve as grouping symbols.

Example 2. Evaluate each of the following expressions.

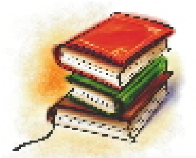
a) $\sqrt{25} - \sqrt{16}$ b) $\sqrt{25 - 16}$ c) $\sqrt{144 + 25}$ d) $\sqrt{144} + \sqrt{25}$

Solution: a) $\sqrt{25} - \sqrt{16} = 5 - 4 = \boxed{1}$

b) $\sqrt{25 - 16} = \sqrt{9} = \boxed{3}$

c) $\sqrt{144 + 25} = \sqrt{169} = \boxed{13}$

d) $\sqrt{144} + \sqrt{25} = 12 + 5 = \boxed{17}$



Sample Problems

1. Simplify each of the following expressions.

   a) $\left(\sqrt{3}\right)^2$ b) $\left(-\sqrt{3}\right)^2$ c) $\left(\sqrt{x}\right)^2$

2. Simplify each of the following expressions.

   a) $\left(\sqrt{5}\right)^2$ b) $\left(\sqrt{3}\right)^4$ c) $\left(\sqrt{2}\right)^{10}$ d) $\left(-\sqrt{2}\right)^{10}$ e) $\left(\sqrt{x}\right)^{12}$

3. Simplify each of the following expressions. Assume that a represents a positive number.

   a) $\sqrt{32}$

   b) $\sqrt{45}$

   c) $\sqrt{48x^5y^3}$

   d) $\sqrt{125}-3\sqrt{80}+\sqrt{45}$

   e) $\dfrac{\sqrt{24}}{\sqrt{54}}$

   f) $\sqrt{80a^{11}}-2\sqrt{180a^{11}}+3\sqrt{245a^{11}}$

   g) $\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)$

   h) $\left(\sqrt{7}-2\right)^2$

   i) $\left(\sqrt{3}-1\right)^3$

   j) $\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right)$

   k) $\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right)$

   l) $\left(\sqrt{x}-\sqrt{2}\right)^2$

4. Rationalize the denominator in each of the following expressions.

   a) $\dfrac{3}{\sqrt{5}}$ b) $\dfrac{1}{\sqrt{10}-3}$ c) $\dfrac{2}{\sqrt{7}+1}$ d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$ e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

5. Simplify each of the following expressions.

   a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$

   b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)$

   c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$

   d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$

   e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$

6. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

7. Solve each of the following quadratic equations by completing the square. Check your solution(s).

   a) $x^2=4x+1$ b) $x^2+13=8x$

Practice Problems

1. Simplify each of the following expressions.

   a) $\left(\sqrt{5}\right)^2$ b) $\left(-\sqrt{5}\right)^2$ c) $\left(\sqrt{y}\right)^2$

2. Simplify each of the following expressions.

   a) $\left(\sqrt{7}\right)^2$ b) $\left(\sqrt{2}\right)^6$ c) $\left(\sqrt{5}\right)^4$ d) $\left(-\sqrt{3}\right)^6$ e) $\left(\sqrt{x}\right)^{100}$

3. Simplify each of the following expressions. Assume that a represents a positive number.

   a) $\sqrt{20}$
   b) $\sqrt{300} - 2\sqrt{75} + \sqrt{12}$
   c) $3\sqrt{20} - 4\sqrt{45} + 2\sqrt{245}$
   d) $5\sqrt{18a^5} - 7\sqrt{32a^5} + 4\sqrt{50a^5}$
   e) $\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)$
   f) $\left(\sqrt{5}-2\right)^2$
   g) $\left(\sqrt{5}-2\right)^3$
   h) $\left(3\sqrt{5}-1\right)\left(7\sqrt{5}+2\right)$

4. Rationalize the denominator in each of the following expressions.

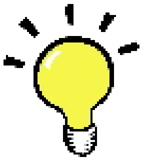
   a) $\dfrac{4}{\sqrt{7}}$
   b) $\dfrac{1}{\sqrt{7}-3}$
   c) $\dfrac{1}{\sqrt{10}+3}$
   d) $\dfrac{2}{\sqrt{17}+4}$
   e) $\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$
   f) $\dfrac{\sqrt{10}+\sqrt{2}}{\sqrt{10}-\sqrt{2}}$
   g) $\dfrac{-5x+5}{\sqrt{x}+1}$
   h) $\dfrac{2}{\sqrt{x}+4}$

5. Find the exact value of

   a) $x^2 - 6x + 1$ if $x = 3 - \sqrt{10}$ b) $b^2 + 8b - 20$ if $b = \sqrt{5} - 2$ c) $x^2 - 10x + 16$ if $x = \sqrt{6} + 5$

6. Solve each of the following quadratic equations by completing the square. Check your solution(s).

   a) $x^2 + 47 = 14x$ b) $x^2 + 12x + 31 = 0$ c) $x^2 = 2x + 1$



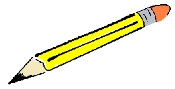
Answers

Sample Problems

1. a) 3 b) 3 c) x
2. a) 5 b) 9 c) 32 d) 32 e) x^6
3. a) $4\sqrt{2}$ b) $3\sqrt{5}$ c) $4x^2y\sqrt{3xy}$ d) $-4\sqrt{5}$ e) $\dfrac{2}{3}$ f) $13a^5\sqrt{5a}$ g) 3
   h) $11-4\sqrt{7}$ i) $-10+6\sqrt{3}$ j) $5x+\sqrt{5x}-6$ k) $6+\sqrt{x}-2x$ l) $x-2\sqrt{2x}+2$
4. a) $\dfrac{3\sqrt{5}}{5}$ b) $3+\sqrt{10}$ c) $\dfrac{-1+\sqrt{7}}{3}$ d) $3+\sqrt{5}$ e) $\dfrac{13-4\sqrt{10}}{3}$
5. a) 7 b) 172 c) 3 d) 4 e) 22
6. 5
7. a) $2-\sqrt{5},\ \ 2+\sqrt{5}$ b) $4-\sqrt{3},\ \ 4+\sqrt{3}$

Practice Problems

1. a) 5 b) 5 c) y
2. a) 7 b) 8 c) 25 d) 27 e) x^{50}
3. a) $2\sqrt{5}$ b) $2\sqrt{3}$ c) $8\sqrt{5}$ d) $7a^2\sqrt{2a}$ e) 1 f) $9-4\sqrt{5}$ g) $17\sqrt{5}-38$ h) $103-\sqrt{5}$
4. a) $\dfrac{4\sqrt{7}}{7}$ b) $-\dfrac{3+\sqrt{7}}{2}$ c) $-3+\sqrt{10}$ d) $-8+2\sqrt{17}$ or $2\left(-4+\sqrt{17}\right)$
   e) $\dfrac{3-\sqrt{5}}{2}$ f) $\dfrac{\sqrt{5}+3}{2}$ g) $-5\sqrt{x}+5$ h) $\dfrac{2\sqrt{x}-8}{x-16}$
5. a) 2 b) $-27+4\sqrt{5}$ c) -3
6. a) $7-\sqrt{2},\ \ 7+\sqrt{2}$ b) $-6-\sqrt{5},\ \ -6+\sqrt{5}$ c) $1-\sqrt{2},\ \ 1+\sqrt{2}$

Sample Problems  Solutions

1. Simplify each of the following expressions.

a) $\left(\sqrt{3}\right)^2$

Solution: By definition, $\left(\sqrt{3}\right)^2 = \boxed{3}$.

b) $\left(-\sqrt{3}\right)^2$

Solution: It does not matter whether we square a negative or a positive number. The result is always positive. Perhaps it helps if we use the rule of exponents $(ab)^n = a^n b^n$.

$$\left(-\sqrt{3}\right)^2 = \left(-1 \cdot \sqrt{3}\right)^2 = (-1)^2 \left(\sqrt{3}\right)^2 = 1 \cdot 3 = \boxed{3}$$

c) $\left(\sqrt{x}\right)^2$

Solution: By definition, $\left(\sqrt{x}\right)^2 = \boxed{x}$.

2. Simplify each of the following expressions.

Solution: a) By definition, $\left(\sqrt{5}\right)^2 = \boxed{5}$.

b) $\left(\sqrt{3}\right)^4$

Solution: To solve this problem, we recall the following rule of exponents: $(a^n)^m = a^{nm}$. By definition,

$$\left(\sqrt{3}\right)^2 = 3. \left(\sqrt{3}\right)^4 = \left(\sqrt{3}\right)^{2 \cdot 2} = \left[\left(\sqrt{3}\right)^2\right]^2 = 3^2 = \boxed{9}$$

c) $\left(\sqrt{2}\right)^{10}$

Solution: To solve this problem, we recall the following rule of exponents: $(a^n)^m = a^{nm}$. By definition, $\left(\sqrt{2}\right)^2 = 2$.

$$\left(\sqrt{2}\right)^{10} = \left(\sqrt{2}\right)^{2 \cdot 5} = \left[\left(\sqrt{2}\right)^2\right]^5 = 2^5 = \boxed{32}$$

d) $\left(-\sqrt{2}\right)^{10}$

Solution: To solve this problem, we recall the following rule of exponents: $(a^n)^m = a^{nm}$. By definition, $\left(\sqrt{2}\right)^2 = 2$.

$$\left(-\sqrt{2}\right)^{10} = \left(-\sqrt{2}\right)^{2 \cdot 5} = \left[\left(-\sqrt{2}\right)^2\right]^5 = 2^5 = \boxed{32}$$

e) $\left(\sqrt{x}\right)^{12}$

Solution: To solve this problem, we recall the following rule of exponents: $(a^n)^m = a^{nm}$. By definition, $\left(\sqrt{x}\right)^2 = x$.

$$\left(\sqrt{x}\right)^{12} = \left(\sqrt{x}\right)^{2 \cdot 6} = \left[\left(\sqrt{x}\right)^2\right]^6 = \boxed{x^6}$$

3. Simplify each of the following expressions. Assume that a represents a positive number.

a) $\sqrt{32}$

Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = \boxed{4\sqrt{2}}$$

b) $\sqrt{45}$
Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\sqrt{5} = \boxed{3\sqrt{5}}$$

c) $\sqrt{48x^5y^3}$
Solution: We first factor the expressions under the square root sign into two factors, where the first factor is the largest square we can find in the expression. Then this first part can come out from under the square root sign.

$$\sqrt{48x^5y^3} = \sqrt{16x^4y^2 \cdot 3xy} = \sqrt{16x^4y^2}\sqrt{3xy} = \boxed{4x^2y\sqrt{3xy}}$$

d) $\sqrt{125} - 3\sqrt{80} + \sqrt{45} =$
Solution: We simplify each expression as in the previous problem. Then we combine like radicals.

$$\begin{aligned}
\sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25 \cdot 5} - 3\sqrt{16 \cdot 5} + \sqrt{9 \cdot 5} \\
&= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5} \\
&= 5\sqrt{5} - 3 \cdot 4 \cdot \sqrt{5} + 3\sqrt{5} \\
&= 5\sqrt{5} - 12\sqrt{5} + 3\sqrt{5} \\
&= (5 - 12 + 3)\sqrt{5} = \boxed{-4\sqrt{5}}
\end{aligned}$$

e) $\dfrac{\sqrt{24}}{\sqrt{54}}$
Solution: We simlify both radical expressions and simplify.

$$\frac{\sqrt{24}}{\sqrt{54}} = \frac{\sqrt{4 \cdot 6}}{\sqrt{9 \cdot 6}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{9}\sqrt{6}} = \frac{2\sqrt{6}}{3\sqrt{6}} = \boxed{\frac{2}{3}}$$

f) $\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}}$
Solution:

$$\begin{aligned}
\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}} &= \\
\sqrt{16a^{10} \cdot 5a} - 2\sqrt{36a^{10} \cdot 5a} + 3\sqrt{49a^{10} \cdot 5a} &= \\
\sqrt{16a^{10}}\sqrt{5a} - 2\sqrt{36a^{10}}\sqrt{5a} + 3\sqrt{49a^{10}}\sqrt{5a} &= \\
4a^5\sqrt{5a} - 2\left(6a^5\right)\sqrt{5a} + 3\left(7a^5\right)\sqrt{5a} &= \\
4a^5\sqrt{5a} - 12a^5\sqrt{5a} + 21a^5\sqrt{5a} &= \\
(4 - 12 + 21)\, a^5\sqrt{5a} &= \boxed{13a^5\sqrt{5a}}
\end{aligned}$$

Note: We can rearrange the final answer as $13a^5\sqrt{5a} = 13\sqrt{5}a^5\sqrt{a}$. This other form is just as correct and might even be preferable in some cases.

g) $\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)$
Solution:

$$\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right) = \sqrt{7}\sqrt{7} - 2\sqrt{7} + 2\sqrt{7} - 4 = 7 - 4 = \boxed{3}$$

h) $\left(\sqrt{7}-2\right)^2$
Solution:

$$\begin{aligned}
\left(\sqrt{7}-2\right)^2 &= \left(\sqrt{7}-2\right)\left(\sqrt{7}-2\right) \\
&= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4 \\
&= 7 - 4\sqrt{7} + 4 = \boxed{11 - 4\sqrt{7}}
\end{aligned}$$

i) $\left(\sqrt{3}-1\right)^3$

Solution: We will first work out $\left(\sqrt{3}-1\right)^2$ and then multiply the result by $\left(\sqrt{3}-1\right)$.

$$\begin{aligned}
\left(\sqrt{3}-1\right)^3 &= \left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right) \\
&= \left(\sqrt{3}\sqrt{3}-1\sqrt{3}-1\sqrt{3}+1\right)\left(\sqrt{3}-1\right) \\
&= \left(3-2\sqrt{3}+1\right)\left(\sqrt{3}-1\right) \\
&= \left(4-2\sqrt{3}\right)\left(\sqrt{3}-1\right) \\
&= 4\sqrt{3}-4-2\sqrt{3}\sqrt{3}+2\sqrt{3} \\
&= 4\sqrt{3}-4-2\cdot 3+2\sqrt{3} \\
&= 4\sqrt{3}-4-6+2\sqrt{3} \\
&= \boxed{-10+6\sqrt{3}}
\end{aligned}$$

j) $\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right)$

Solution:

$$\begin{aligned}
\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right) &= \sqrt{5x}\sqrt{5x}+3\sqrt{5x}-2\sqrt{5x}-2\cdot 3 = 5x+3\sqrt{5x}-2\sqrt{5x}-6 \\
&= \boxed{5x+\sqrt{5x}-6}
\end{aligned}$$

k) $\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right)$

Solution:

$$\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right) = 6+4\sqrt{x}-3\sqrt{x}-2\sqrt{x}\sqrt{x} = \boxed{6+\sqrt{x}-2x}$$

l) $\left(\sqrt{x}-\sqrt{2}\right)^2$

Solution:

$$\begin{aligned}
\left(\sqrt{x}-\sqrt{2}\right)^2 &= \left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}-\sqrt{2}\right) && \text{we FOIL} \\
&= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{2}-\sqrt{2}\sqrt{x}+\sqrt{2}\sqrt{2} \\
&= x-\sqrt{2x}-\sqrt{2x}+2 = \boxed{x-2\sqrt{2x}+2}
\end{aligned}$$

4. Rationalize the denominator in each of the following expressions.

a) $\dfrac{3}{\sqrt{5}}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{5}}$$

b) $\dfrac{1}{\sqrt{10}-3}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3} = \frac{1}{\sqrt{10}-3}\cdot\frac{\sqrt{10}+3}{\sqrt{10}+3} = \frac{\sqrt{10}+3}{1} = \boxed{3+\sqrt{10}}$$

The denominator is 1 since

$$\left(\sqrt{10}-3\right)\left(\sqrt{10}+3\right) = \sqrt{10}\sqrt{10}+3\sqrt{10}-3\sqrt{10}-9 = 10-9 = 1$$

c) $\dfrac{2}{\sqrt{7}+1}$
Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1}=\frac{2}{\sqrt{7}+1}\cdot\frac{\sqrt{7}-1}{\sqrt{7}-1}=\frac{2\left(\sqrt{7}-1\right)}{7-1}=\frac{2\left(\sqrt{7}-1\right)}{6}=\boxed{\frac{\sqrt{7}-1}{3}}$$

d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$
Solution:

$$\frac{\sqrt{5}-1}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\left(\sqrt{5}-1\right)\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)}=\frac{5+2\sqrt{5}-\sqrt{5}-2}{5+2\sqrt{5}-2\sqrt{5}-4}=\frac{3+\sqrt{5}}{1}=\boxed{3+\sqrt{5}}$$

e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$
Solution: We multiply the fraction by 1 as a fraction of whose both numerator and denominator are the conjugate of the denominator.

$$\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot 1=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}-\sqrt{5}}=\frac{\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}{\left(\sqrt{8}+\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}$$

We FOIL out both numerator and denominator

$$\frac{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}-\sqrt{5}\sqrt{8}+\sqrt{5}\sqrt{5}}{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}+\sqrt{5}\sqrt{8}-\sqrt{5}\sqrt{5}}=\frac{8-\sqrt{40}-\sqrt{40}+5}{8-5}=\frac{13-2\sqrt{40}}{3}$$

Note: although this answer is acceptable, the expression can be further simplified.

$$\frac{13-2\sqrt{40}}{3}=\frac{13-2\sqrt{4\cdot 10}}{3}=\frac{13-2\sqrt{4}\cdot\sqrt{10}}{3}=\frac{13-2\cdot 2\cdot\sqrt{10}}{3}=\boxed{\frac{13-4\sqrt{10}}{3}}$$

5. Simplify each of the following expressions.

a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$
Solution:

$$\frac{3-\sqrt{5}}{3+\sqrt{5}}+\frac{3+\sqrt{5}}{3-\sqrt{5}}=\frac{\left(3-\sqrt{5}\right)^2+\left(3+\sqrt{5}\right)^2}{\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)}=\frac{14-6\sqrt{5}+14+6\sqrt{5}}{9-5}=\frac{28}{4}=\boxed{7}$$

b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)$
Solution: We first bring the fractions to the common denominator - they are conjugates, that helps, because the common denominator is then rational. We then multiply the factors.

$$\left(\frac{8}{\sqrt{7}+\sqrt{3}}+\frac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)=$$

$$\begin{aligned}
&=\frac{8\left(\sqrt{7}-\sqrt{3}\right)+12\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)}\left(5\sqrt{7}-\sqrt{3}\right)\\
&=\frac{8\sqrt{7}-8\sqrt{3}+12\sqrt{7}+12\sqrt{3}}{7-3}\left(5\sqrt{7}-\sqrt{3}\right)\\
&=\frac{20\sqrt{7}+4\sqrt{3}}{4}\left(5\sqrt{7}-\sqrt{3}\right)=\left(5\sqrt{7}+\sqrt{3}\right)\left(5\sqrt{7}-\sqrt{3}\right)\\
&=25\,(7)-3=175-3=\boxed{172}
\end{aligned}$$

c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$
Solution:

$$\begin{aligned}\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}} &= \sqrt{\sqrt{41}+\sqrt{32}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}=\sqrt{\left(\sqrt{41}+\sqrt{32}\right)\left(\sqrt{41}-\sqrt{32}\right)}\\ &= \sqrt{\left(\left(\sqrt{41}\right)^2-\left(\sqrt{32}\right)^2\right)}=\sqrt{41-32}=\sqrt{9}=\boxed{3}\end{aligned}$$

d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$
Solution:

$$\begin{aligned}\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}} &= \sqrt{\sqrt{75}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}=\sqrt{\left(\sqrt{75}+\sqrt{59}\right)\left(\sqrt{75}-\sqrt{59}\right)}\\ &= \sqrt{\left(\left(\sqrt{75}\right)^2-\left(\sqrt{59}\right)^2\right)}=\sqrt{75-59}=\sqrt{16}=\boxed{4}\end{aligned}$$

e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$
Solution:

$$\begin{aligned}\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2 &= \left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}}\\ &= 6+\sqrt{11}+6-\sqrt{11}+2\sqrt{\left(6+\sqrt{11}\right)\left(6-\sqrt{11}\right)}\\ &= 12+2\sqrt{36-11}=12+2\sqrt{25}=12+2\cdot 5=\boxed{22}\end{aligned}$$

Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.
Solution: We work out x^2 first.

$$\begin{aligned}x^2 &= \left(2-\sqrt{3}\right)^2=\left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)\\ &= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3}\\ &= 4-4\sqrt{3}+3=7-4\sqrt{3}\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6.

$$\begin{aligned}x^2-4x+6 &= \left(2-\sqrt{3}\right)^2-4\left(2-\sqrt{3}\right)+6=\\ &= 7-4\sqrt{3}-8+4\sqrt{3}+6=7-8+6=\boxed{5}\end{aligned}$$

6. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2=4x+1$
Solution: We factor by completing the square.

$$\begin{aligned}x^2 &= 4x+1 && \text{reduce one side to zero}\\ x^2-4x-1 &= 0 && (x-2)^2=x^2-4x+4\\ \underbrace{x^2-4x+4}-4-1 &= 0\\ (x-2)^2-5 &= 0\\ (x-2)^2-\left(\sqrt{5}\right)^2 &= 0\\ \left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right) &= 0\end{aligned}$$

$$\boxed{x_1 = 2-\sqrt{5} \quad \text{and} \quad x_2 = 2+\sqrt{5}}$$

We check: if $x = 2-\sqrt{5}$, then

$$\begin{aligned}
\text{LHS} &= \left(2-\sqrt{5}\right)^2 = \left(2-\sqrt{5}\right)\left(2-\sqrt{5}\right) = 4-2\sqrt{5}-2\sqrt{5}+\sqrt{5}\sqrt{5} = 4-4\sqrt{5}+5 = 9-4\sqrt{5} \\
\text{RHS} &= 4\left(2-\sqrt{5}\right)+1 = 8-4\sqrt{5}+1 = 9-4\sqrt{5}
\end{aligned}$$

and if $x = 2+\sqrt{5}$, then

$$\begin{aligned}
\text{LHS} &= \left(2+\sqrt{5}\right)^2 = \left(2+\sqrt{5}\right)\left(2+\sqrt{5}\right) = 4+2\sqrt{5}+2\sqrt{5}+\sqrt{5}\sqrt{5} = 4+4\sqrt{5}+5 = 9+4\sqrt{5} \\
\text{RHS} &= 4\left(2+\sqrt{5}\right)+1 = 8+4\sqrt{5}+1 = 9+4\sqrt{5}
\end{aligned}$$

So, both solutions are correct.

b) $x^2+13 = 8x$

Solution: We factor by completing the square.

$$\begin{aligned}
x^2-8x+13 &= 0 \qquad\qquad (x-4)^2 = x^2-8x+16 \\
\underbrace{x^2-8x+16}-16+13 &= 0 \\
(x-4)^2-3 &= 0 \\
(x-4)^2-\left(\sqrt{3}\right)^2 &= 0 \\
\left(x-4+\sqrt{3}\right)\left(x-4-\sqrt{3}\right) &= 0
\end{aligned}$$

$$\boxed{x_1 = 4-\sqrt{3} \ \text{ and } \ x_2 = 4+\sqrt{3}}$$

We check: if $x = 4-\sqrt{3}$, then

$$\begin{aligned}
\text{LHS} &= \left(4-\sqrt{3}\right)^2+13 = \left(4-\sqrt{3}\right)\left(4-\sqrt{3}\right)+13 = 16-4\sqrt{3}-4\sqrt{3}+3+13 = 32-8\sqrt{3} \\
\text{RHS} &= 8\left(4-\sqrt{3}\right) = 32-8\sqrt{3}
\end{aligned}$$

and if $x = 4+\sqrt{3}$, then

$$\begin{aligned}
\text{LHS} &= \left(4+\sqrt{3}\right)^2+13 = \left(4+\sqrt{3}\right)\left(4+\sqrt{3}\right)+13 = 16+4\sqrt{3}+4\sqrt{3}+3+13 = 32+8\sqrt{3} \\
\text{RHS} &= 8\left(4+\sqrt{3}\right) = 32+8\sqrt{3}
\end{aligned}$$

So, both solutions are correct.

