Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

> **Definition**: Let N be a non-negative number. Then the **square root of** N (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N. If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

Square roots, when stretched over entire expressions, also serve as grouping symbols.

Example 2. Evaluate each of the following expressions.

a) $\sqrt{25} - \sqrt{16}$ b) $\sqrt{25 - 16}$ c) $\sqrt{144 + 25}$ d) $\sqrt{144} + \sqrt{25}$

Solution: a) $\sqrt{25} - \sqrt{16} = 5 - 4 = \boxed{1}$

b) $\sqrt{25 - 16} = \sqrt{9} = \boxed{3}$

c) $\sqrt{144 + 25} = \sqrt{169} = \boxed{13}$

d) $\sqrt{144} + \sqrt{25} = 12 + 5 = \boxed{17}$

Not all square roots are as nice as $\sqrt{9}$ or $\sqrt{16}$. In a previous section we have seen that $\sqrt{2}$ is a real number but not rational.

> **Definition**: A number is **rational** if it can be written as a quotient of two integers. A number is **irrational** if it can not be written as a quotient of two integers.

The set of all real numbers is the union of rational and irrational numbers. There are many types of irrational numbers, but our first example was $\sqrt{2}$. If a non-negative number is not a perfect square, then its square root exists but is an irrational real number. So, all of $\sqrt{3}$, $\sqrt{8}$, and $\sqrt{10}$ are also irrational.

So how can we deal with numbers like $\sqrt{2}$? By definition, irrational numbers can not be written as a fraction of integers. We have also seen before that irrational numbers have terrible decimal presentations: they are non-terminating and non-repeating. Therefore, we can not write the exact value of $\sqrt{2}$ as a decimal either: all attempts to do so would result in *approximations* as opposed to *exact value*.

In order to write the exact value of $\sqrt{2}$, we can not use fractions or decimals. Our only tool is to write it as a radical, as $\sqrt{2}$. This is an elusive positive real number that, when we square, the result is 2. Our only option is to include these radical expressions in our computations. Until now we had numbers and letters. From now on, we will have nice numbers (i.e rational numbers), yucky numbers (i.e. irrational numbers), and the variables. In what follows, we will explore how we can compute with these new radical expressions in our algebra.

By definition, $\sqrt{2}$ is the unique positive number that, when squared, the result is 2. Therefore,

$$\left(\sqrt{2}\right)^2 = 2$$

One useful skill is to quickly estimate the value of a radical.

Example 3. Estimate each of the following radical expressions by placing them between two consecutive integers.

a) $\sqrt{20}$ b) $\sqrt{50}$ c) $\sqrt{5}$

Solution: a) If we are looking at positive numbers only, the greater the number, the greater is its square. In order to estimate $\sqrt{20}$ we need to find two perfect squares, one less than 20, and the other greater than 20. In this case, 16 and 25 are two consecutive perfect squares such that 20 is locked between them.

$$16 < 20 < 25, \text{ therefore } 4 < \sqrt{20} < 5$$

Our conclusion is that $\sqrt{20}$ is a number between 4 and 5. We can check this using our calculator: $\sqrt{20} \approx 4.4721$.

b) We need to find two consecutive squares such that 50 is between them. Since $49 < 50 < 64$, we have that $7 < \sqrt{50} < 8$ that $\sqrt{50}$ is a number between 7 and 8.

c) Since 5 is a number between 4 and 9, we conclude that $\sqrt{5}$ is a number between 2 and 3.

Square roots are defined in terms of squaring, which is multiplication. As a result, radical expressions will have nice properties with respect to multiplication and division, but not addition and subtraction.

Theorem: For all positive numbers a and b,

$$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \qquad \text{but} \quad \sqrt{a}+\sqrt{b} \neq \sqrt{a+b} \quad \text{and} \quad \sqrt{a}-\sqrt{b} \neq \sqrt{a-b}$$

We can see these properties using perfect squares.

Example 4. Evaluate each of the folllowing expressions.

a) $\sqrt{4}\sqrt{9}$ b) $\sqrt{4\cdot 9}$ c) $\dfrac{\sqrt{100}}{\sqrt{25}}$ d) $\sqrt{\dfrac{100}{25}}$

e) $\sqrt{9}+\sqrt{16}$ f) $\sqrt{9+16}$ g) $\sqrt{25}-\sqrt{9}$ h) $\sqrt{25-9}$

Solution: a) $\sqrt{4}\sqrt{9} = 2\cdot 3 = \boxed{6}$

b) $\sqrt{4\cdot 9} = \sqrt{36} = \boxed{6}$

c) $\dfrac{\sqrt{100}}{\sqrt{25}} = \dfrac{10}{5} = \boxed{2}$

d) $\sqrt{\dfrac{100}{25}} = \sqrt{4} = \boxed{2}$

e) $\sqrt{9}+\sqrt{16} = 3+4 = \boxed{7}$

f) $\sqrt{9+16} = \sqrt{25} = \boxed{5}$

g) $\sqrt{25}-\sqrt{9} = 5-3 = \boxed{2}$

h) $\sqrt{25-9} = \sqrt{16} = \boxed{4}$

One fundamental property of square roots is the result after squaring. Irrational numbers are elusive, but one thing we know about $\sqrt{5}$ is that once we square it, the result is 5. Also recall the rule $(a^n)^m = a^{nm}$.

Example 5. Evaluate each of the following expressions.

a) $\left(\sqrt{7}\right)^2$ b) $\left(\sqrt{2}\right)^6$ c) $\left(-\sqrt{3}\right)^2$ d) $\sqrt{2}\cdot\sqrt{18}$ e) $\left(2\sqrt{5}\right)^2$

Solution: a) By definition of square roots, $\left(\sqrt{7}\right)^2 = 7$.

b) We know that $\left(\sqrt{2}\right)^2 = 2$. We will use the rule $(a^n)^m = a^{nm}$ to extract a 2 from the exponent.

$$\left(\sqrt{2}\right)^6 = \left(\sqrt{2}\right)^{2\cdot 3} = \left[\left(\sqrt{2}\right)^2\right]^3 = 2^3 = \boxed{8}$$

c) $\left(-\sqrt{3}\right)^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) = \left(\sqrt{3}\right)^2 = \boxed{3}$

d) We will use the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

$$\sqrt{2}\cdot\sqrt{18} = \sqrt{2\cdot 18} = \sqrt{36} = \boxed{6}$$

e) Recall the rule of exponents $(ab)^n = a^n b^n$.

$$\left(2\sqrt{5}\right)^2 = 2^2\left(\sqrt{5}\right)^2 = 4\cdot 5 = \boxed{20}$$

This last example suggests that $2\sqrt{5}$ and $\sqrt{20}$ are the same number, because they are both positive and when squared, the result is 20. Indeed, that is true, $\sqrt{20} = 2\sqrt{5}$. We can use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to verify this and also, to simplify more complicated square roots.

Example 6. Simplify each of the following radicals.

a) $\sqrt{20}$ b) $\sqrt{50}$ c) $\sqrt{72}$ d) $\sqrt{12}$

Solution: a) We will express 20 as a product in which one factor is a square, the greatest square factor of 20 that we can find. Then we separate the square part from the rest via the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and simplify.

$$\sqrt{20} = \sqrt{4\cdot 5} = \sqrt{4}\sqrt{5} = \boxed{2\sqrt{5}}$$

b) $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\sqrt{2} = \boxed{5\sqrt{2}}$

We can check that this is true by squaring both sides. $\left(\sqrt{50}\right)^2 = 50$ by definition of square roots and $\left(5\sqrt{2}\right)^2 = 5^2\left(\sqrt{2}\right)^2 = 25\cdot 2 = 50$.

c) $\sqrt{72} = \sqrt{36\cdot 2} = \sqrt{36}\sqrt{2} = \boxed{6\sqrt{2}}$

d) $\sqrt{12} = \sqrt{4\cdot 3} = \sqrt{4}\sqrt{3} = \boxed{2\sqrt{3}}$

Consider the last example, $\sqrt{12} = 2\sqrt{3}$. These two are equal, and officially, $\sqrt{12}$ is not simplified. Theoretically, a number such as $\sqrt{12}$ must be presented in its simplified form, as $2\sqrt{3}$. This does not necessarily mean that the form $2\sqrt{3}$ is always better than $\sqrt{12}$. Depending on the computation, sometimes one form is more suitable than the other. We will see later that both forms have their own advantages.

Now we will begin to perform operations that involve radical expressions. When it comes to addition and subtraction, radicals are unlike terms to rational numbers. Their behavior is like that of a variable. Just as much as $3x + 2$ cannot be simplified, $3\sqrt{5} + 2$ can also not be simplified. Just as much as $3x + 4x$ can be simplified to $7x$, $3\sqrt{5} + 4\sqrt{5}$ can be simplified. Worse yet, $\sqrt{2}$ and $\sqrt{3}$ are also unlike terms to each other, like the variables x and y. Two radicals can only be combined if they are the square root of the same number.

Example 7. Simplify each of the following expressions.

a) $\sqrt{3}-2+7\sqrt{3}+4$ b) $\sqrt{50}-3\sqrt{8}$ c) $3\sqrt{5}-\sqrt{5}+\sqrt{20}$

Solution: a) $\sqrt{7}$ and $7\sqrt{3}$ are like terms and -2 and 4 are like terms, just like in the expression $x-2+7x+4$. If we use this analogy, $x-2+7x+4=(1x+7x)+(-2+4)=8x+2$. Similarly,

$$\sqrt{3}-2+7\sqrt{3}+4=\boxed{8\sqrt{3}+2}$$

b) We will first simplify the radicals to see if they can be combined.

$$\sqrt{50}=\sqrt{25\cdot 2}=\sqrt{25}\sqrt{2}=5\sqrt{2} \text{ and } \sqrt{8}=\sqrt{4\cdot 2}=\sqrt{4}\sqrt{2}=2\sqrt{2}$$

These expressions are like terms and so they can be combined:

$$\sqrt{50}-3\sqrt{8}=5\sqrt{2}-3\left(2\sqrt{6}\right)=5\sqrt{2}-6\sqrt{2}=-1\sqrt{2}=\boxed{-\sqrt{2}}$$

c) We simplify $\sqrt{20}$ first, to see if it can be combined with the other terms.

$$\sqrt{20}=\sqrt{4\cdot 5}=\sqrt{4}\sqrt{5}=2\sqrt{5}$$

Now we can combine all three terms:

$$3\sqrt{5}-\sqrt{5}+\sqrt{20}=3\sqrt{5}-\sqrt{5}+2\sqrt{5}=(3-1+2)\sqrt{5}=\boxed{4\sqrt{2}}$$

As our algebraic language is growing to include radicals, we pay attention to preserve the rules we already have. One important example is the distributive law.

Example 8. Perform each of the multiplications as indicated.

a) $\sqrt{2}\left(3\sqrt{2}-1\right)$ b) $\sqrt{3}\left(4-\sqrt{12}\right)$ c) $\sqrt{5}\left(3+\sqrt{2}\right)$

Solution: a) We apply the distributive law.

$$\sqrt{2}\left(3\sqrt{2}-1\right)=\sqrt{2}\cdot 3\sqrt{2}-\sqrt{2}\cdot 1=3\left(\sqrt{2}\right)^2-\sqrt{2}=3\cdot 2-\sqrt{2}=\boxed{6-\sqrt{2}}$$

b) $\sqrt{3}\left(4-\sqrt{12}\right)=\sqrt{3}\cdot 4-\sqrt{3}\sqrt{12}=4\sqrt{3}-\sqrt{3\cdot 12}=4\sqrt{3}-\sqrt{36}=\boxed{4\sqrt{3}-6}$

c) $\sqrt{5}\left(3+\sqrt{2}\right)=\sqrt{5}\cdot 3+\sqrt{5}\sqrt{2}=3\sqrt{5}+\sqrt{5\cdot 2}=3\sqrt{5}+\sqrt{10}=\boxed{3\sqrt{5}+\sqrt{10}}$

As a special case of the distributive law, when we multiply two binomial expressions, FOIL still applies, where F is for first with first, O for outer terms, I for inner terms, and L for last term with last term.

Example 9. Perform each of the multiplications as indicated.

a) $\left(\sqrt{3}-2\right)\left(2\sqrt{3}+1\right)$ b) $\left(\sqrt{2}-1\right)\left(5\sqrt{2}-3\right)$ c) $\left(\sqrt{5}-1\right)\left(3\sqrt{5}+2\right)$

Solution: a) We apply the distributive law. Consider the multiplication $(x-2)(2x+1)$. When we work this out, we get

$$\begin{aligned}(x-2)(2x+1)&=\overset{\text{F}}{x\cdot 2x}+\overset{\text{O}}{x\cdot 1}-\overset{\text{I}}{2\cdot 2x}-\overset{\text{L}}{2\cdot 1}\\&=2x^2+x-4x-2 && \text{combine like terms O and I}\\&=2x^2-3x-2\end{aligned}$$

This situation is similar to ours, as $\sqrt{3}$ behaves like x does, with one exception: that its square is 3 and not x^2.

$$\begin{aligned}
\left(\sqrt{3}-2\right)\left(2\sqrt{3}+1\right) &= \overset{\text{F}}{\sqrt{3}\cdot 2\sqrt{3}} + \overset{\text{O}}{\sqrt{3}\cdot 1} - \overset{\text{I}}{2\cdot 2\sqrt{3}} - \overset{\text{L}}{2\cdot 1} \\
&= 2\left(\sqrt{3}\right)^2 + \sqrt{3} - 4\sqrt{3} - 2 \\
&= 2\cdot 3 + \sqrt{3} - 4\sqrt{3} - 2 \\
&= 6 - 3\sqrt{3} - 2 = \boxed{4-3\sqrt{3}}
\end{aligned}$$

b) $$\begin{aligned}
\left(\sqrt{2}-1\right)\left(5\sqrt{2}-3\right) &= \overset{\text{F}}{\sqrt{2}\cdot 5\sqrt{2}} - \overset{\text{O}}{\sqrt{2}\cdot 3} - \overset{\text{I}}{1\cdot 5\sqrt{2}} + \overset{\text{L}}{1\cdot 3} \\
&= 5\left(\sqrt{2}\right)^2 - 3\sqrt{2} - 5\sqrt{2} + 3 \\
&= 5\cdot 2 - 8\sqrt{2} + 3 \\
&= 10 - 8\sqrt{2} + 3 = \boxed{13-8\sqrt{2}}
\end{aligned}$$

c) The previous examples showed many details so that the reader can understand. In this example, we will show as much notation as is needed to be shown by students. Writing too much can make reading computations as difficult as not writing enough.

$$\begin{aligned}
\left(\sqrt{5}-1\right)\left(3\sqrt{5}+2\right) &= 3\left(\sqrt{5}\right)^2 + 2\sqrt{5} - 3\sqrt{5} - 2 \\
&= 3\cdot 5 - \sqrt{5} - 2 \\
&= 15 - \sqrt{5} - 2 = \boxed{13-\sqrt{5}}
\end{aligned}$$

The following examples show special products such as complete squares. Notice that if we square an expression such as $2\sqrt{5}-1$, the result will still have radical expressions in it.

Example 10. Perform each of the multiplications as indicated.

a) $\left(2\sqrt{5}-1\right)^2$ b) $\left(\sqrt{3}-2\right)^3$ c) $\left(\sqrt{5}-\sqrt{2}\right)^2$

Solution: a) $$\begin{aligned}
\left(2\sqrt{5}-1\right)^2 &= \left(2\sqrt{5}-1\right)\left(2\sqrt{5}-1\right) \\
&= \left(2\sqrt{5}\right)^2 - 2\sqrt{5} - 2\sqrt{5} + 1 \\
&= 2^2\left(\sqrt{5}\right)^2 - 4\sqrt{5} + 1 \\
&= 4\cdot 5 - 4\sqrt{5} + 1 = 20 - 4\sqrt{5} + 1 = \boxed{21-4\sqrt{5}}
\end{aligned}$$

b) We will first work out $\left(\sqrt{3}-2\right)^2$ and then $\left(\sqrt{3}-2\right)^3$.

$$\begin{aligned}
\left(\sqrt{3}-2\right)^2 &= \left(\sqrt{3}-2\right)\left(\sqrt{3}-2\right) \\
&= \left(\sqrt{3}\right)^2 - 2\sqrt{3} - 2\sqrt{3} + 4 \\
&= 3 - 4\sqrt{3} + 4 = \boxed{7-4\sqrt{3}}
\end{aligned}$$

We are now ready to work out the third power:

$$\begin{aligned}\left(\sqrt{3}-2\right)^3 &= \left(\sqrt{3}-2\right)\left(\sqrt{3}-2\right)^2\\ &= \left(\sqrt{3}-2\right)\left(7-4\sqrt{3}\right)\\ &= \sqrt{3}\cdot 7-4\left(\sqrt{3}\right)^2-2\cdot 7+8\sqrt{3}\\ &= 7\sqrt{3}-4\cdot 3-14+8\sqrt{3}\\ &= 7\sqrt{3}-12-14+8\sqrt{3}\\ &= \boxed{15\sqrt{3}-26}\end{aligned}$$

c) $$\begin{aligned}\left(\sqrt{5}-\sqrt{2}\right)^2 &= \left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}-\sqrt{2}\right)\\ &= \left(\sqrt{5}\right)^2-\sqrt{5}\sqrt{2}-\sqrt{2}\sqrt{5}+\left(\sqrt{2}\right)^2\\ &= 5-\sqrt{10}-\sqrt{10}+2\\ &= \boxed{7-2\sqrt{10}}\end{aligned}$$

The previous examples show that if we want to get rid of radicals in a sum or difference such as $2\sqrt{5}-3$, squaring the expression will not eliminate the radicals. However, conjugates do. Whenever we multiply two conjugates, O and I cancel out each other, and we are left with F and L. This is the difference of squares theorem.

Example 11. Perform each of the multiplications as indicated.

a) $\left(2\sqrt{7}-3\right)\left(2\sqrt{7}+3\right)$ b) $\left(3\sqrt{2}+2\sqrt{6}\right)\left(3\sqrt{2}-2\sqrt{6}\right)$ c) $\left(\sqrt{10}+3\right)\left(\sqrt{10}-3\right)$

Solution: a) $$\begin{aligned}\left(2\sqrt{7}-3\right)\left(2\sqrt{7}+3\right) &= 2^2\left(\sqrt{7}\right)^2+6\sqrt{7}-6\sqrt{7}-9\\ &= 4\cdot 7-9 = 28-9 = \boxed{19}\end{aligned}$$

b) $$\begin{aligned}\left(3\sqrt{2}+2\sqrt{6}\right)\left(3\sqrt{2}-2\sqrt{6}\right) &= \left(3\sqrt{2}\right)^2-\left(3\sqrt{2}\right)\left(2\sqrt{6}\right)+\left(3\sqrt{2}\right)\left(2\sqrt{6}\right)-\left(2\sqrt{6}\right)^2\\ &= 9\cdot 2-4\cdot 6 = 18-24 = \boxed{-6}\end{aligned}$$

c) $$\begin{aligned}\left(\sqrt{10}+3\right)\left(\sqrt{10}-3\right) &= \left(\sqrt{10}\right)^2+3\sqrt{10}-3\sqrt{10}-9\\ &= 10-9 = \boxed{1}\end{aligned}$$

This last example shows that $\sqrt{10}+3$ and $\sqrt{10}-3$ are not only conjugates of each other, but also reciprocals. (Recall that x and y are reciprocals if their product is 1). We will see that this new algebra that now includes radical expressions is very rich in connections.

Sample Problems

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{32}$
2. $\sqrt{45}$
3. $\sqrt{48x^5y^3}$
4. $\sqrt{125}-3\sqrt{80}+\sqrt{45}$
5. $\dfrac{\sqrt{24}}{\sqrt{54}}$
6. $\sqrt{80a^{11}}-2\sqrt{180a^{11}}+3\sqrt{245a^{11}}$
7. $\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)$
8. $\left(\sqrt{7}-2\right)^2$
9. $\left(\sqrt{3}-1\right)^3$
10. $\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right)$
11. $(2-\sqrt{x})(3+2\sqrt{x})$
12. $\left(\sqrt{x}-\sqrt{2}\right)^2$
13. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

Practice Problems

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{20}$
2. $\sqrt{300}-2\sqrt{75}+\sqrt{12}$
3. $3\sqrt{20}-4\sqrt{45}+2\sqrt{245}$
4. $5\sqrt{18a^5}-7\sqrt{32a^5}+4\sqrt{50a^5}$
5. $\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)$
6. $\left(\sqrt{5}-2\right)^2$
7. $\left(\sqrt{5}-2\right)^3$
8. $\left(3\sqrt{5}-1\right)\left(7\sqrt{5}+2\right)$
9. Find the exact value of each of the following.

   a) x^2-6x+1 if $x=3-\sqrt{10}$

   b) $b^2+8b-20$ if $b=\sqrt{5}-2$

   c) $x^2-10x+16$ if $x=\sqrt{6}+5$

Answers

Sample Problems

1. $4\sqrt{2}$ 2. $3\sqrt{5}$ 3. $4x^2y\sqrt{3xy}$ 4. $-4\sqrt{5}$ 5. $\dfrac{2}{3}$ 6. $13a^5\sqrt{5a}$ 7. 3

8. $11-4\sqrt{7}$ 9. $-10+6\sqrt{3}$ 10. $5x+\sqrt{5x}-6$ 11. $6+\sqrt{x}-2x$ 12. $x-2\sqrt{2x}+2$

13. 5

Practice Problems

1. $2\sqrt{5}$ 2. $2\sqrt{3}$ 3. $8\sqrt{5}$ 4. $7a^2\sqrt{2a}$ 5. 1 6. $9-4\sqrt{5}$ 7. $17\sqrt{5}-38$ 8. $103-\sqrt{5}$

9. a) 2 b) $-27+4\sqrt{5}$ c) -3

Sample Problems Solutions

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{32}$

   Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = \boxed{4\sqrt{2}}$$

2. $\sqrt{45}$

   Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\sqrt{5} = \boxed{3\sqrt{5}}$$

3. $\sqrt{48x^5y^3}$

   Solution: We first factor the expressions under the square root sign into two factors, where the first factor is the largest square we can find in the expression. Then this first part can come out from under the square root sign.

$$\sqrt{48x^5y^3} = \sqrt{16x^4y^2 \cdot 3xy} = \sqrt{16x^4y^2}\sqrt{3xy} = \boxed{4x^2y\sqrt{3xy}}$$

4. $\sqrt{125} - 3\sqrt{80} + \sqrt{45}$

   Solution: We simplify each expression as in the previous problem. Then we combine like radicals.

$$\begin{aligned}
\sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25 \cdot 5} - 3\sqrt{16 \cdot 5} + \sqrt{9 \cdot 5} \\
&= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5} \\
&= 5\sqrt{5} - 3 \cdot 4 \cdot \sqrt{5} + 3\sqrt{5} \\
&= 5\sqrt{5} - 12\sqrt{5} + 3\sqrt{5} \\
&= (5 - 12 + 3)\sqrt{5} = \boxed{-4\sqrt{5}}
\end{aligned}$$

5. $\dfrac{\sqrt{24}}{\sqrt{54}}$

   Solution: We simlify both radical expressions and simplify.

$$\frac{\sqrt{24}}{\sqrt{54}} = \frac{\sqrt{4 \cdot 6}}{\sqrt{9 \cdot 6}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{9}\sqrt{6}} = \frac{2\sqrt{6}}{3\sqrt{6}} = \boxed{\frac{2}{3}}$$

6. $\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}}$

Solution:

$$\begin{aligned}
\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}} &= \\
\sqrt{16a^{10} \cdot 5a} - 2\sqrt{36a^{10} \cdot 5a} + 3\sqrt{49a^{10} \cdot 5a} &= \\
\sqrt{16a^{10}}\sqrt{5a} - 2\sqrt{36a^{10}}\sqrt{5a} + 3\sqrt{49a^{10}}\sqrt{5a} &= \\
4a^5\sqrt{5a} - 2\left(6a^5\right)\sqrt{5a} + 3\left(7a^5\right)\sqrt{5a} &= \\
4a^5\sqrt{5a} - 12a^5\sqrt{5a} + 21a^5\sqrt{5a} &= \\
(4 - 12 + 21)\, a^5\sqrt{5a} &= \boxed{13a^5\sqrt{5a}}
\end{aligned}$$

Note: We can rearrange the final answer as $13a^5\sqrt{5a} = 13\sqrt{5}a^5\sqrt{a}$. This other form is just as correct and might even be preferable in some cases.

7. $\left(\sqrt{7} + 2\right)\left(\sqrt{7} - 2\right)$

Solution:

$$\left(\sqrt{7} + 2\right)\left(\sqrt{7} - 2\right) = \sqrt{7}\sqrt{7} - 2\sqrt{7} + 2\sqrt{7} - 4 = 7 - 4 = \boxed{3}$$

8. $\left(\sqrt{7} - 2\right)^2$

Solution:

$$\begin{aligned}
\left(\sqrt{7} - 2\right)^2 &= \left(\sqrt{7} - 2\right)\left(\sqrt{7} - 2\right) \\
&= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4 \\
&= 7 - 4\sqrt{7} + 4 = \boxed{11 - 4\sqrt{7}}
\end{aligned}$$

9. $\left(\sqrt{3} - 1\right)^3$

Solution: We will first work out $\left(\sqrt{3} - 1\right)^2$ and then multiply the result by $\left(\sqrt{3} - 1\right)$.

$$\begin{aligned}
\left(\sqrt{3} - 1\right)^3 &= \left(\sqrt{3} - 1\right)\left(\sqrt{3} - 1\right)\left(\sqrt{3} - 1\right) \\
&= \left(\sqrt{3}\sqrt{3} - 1\sqrt{3} - 1\sqrt{3} + 1\right)\left(\sqrt{3} - 1\right) \\
&= \left(3 - 2\sqrt{3} + 1\right)\left(\sqrt{3} - 1\right) \\
&= \left(4 - 2\sqrt{3}\right)\left(\sqrt{3} - 1\right) \\
&= 4\sqrt{3} - 4 - 2\sqrt{3}\sqrt{3} + 2\sqrt{3} \\
&= 4\sqrt{3} - 4 - 2 \cdot 3 + 2\sqrt{3} \\
&= 4\sqrt{3} - 4 - 6 + 2\sqrt{3} \\
&= \boxed{-10 + 6\sqrt{3}}
\end{aligned}$$

10. $\left(\sqrt{5x} - 2\right)\left(\sqrt{5x} + 3\right)$

Solution:

$$\begin{aligned}
\left(\sqrt{5x} - 2\right)\left(\sqrt{5x} + 3\right) &= \sqrt{5x}\sqrt{5x} + 3\sqrt{5x} - 2\sqrt{5x} - 2 \cdot 3 = 5x + 3\sqrt{5x} - 2\sqrt{5x} - 6 \\
&= \boxed{5x + \sqrt{5x} - 6}
\end{aligned}$$

11. $\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right)$

Solution:

$$\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right)=6+4\sqrt{x}-3\sqrt{x}-2\sqrt{x}\sqrt{x}=\boxed{6+\sqrt{x}-2x}$$

12. $\left(\sqrt{x}-\sqrt{2}\right)^2$

Solution:

$$\begin{aligned}\left(\sqrt{x}-\sqrt{2}\right)^2 &= \left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}-\sqrt{2}\right) \qquad \text{we FOIL}\\ &= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{2}-\sqrt{2}\sqrt{x}+\sqrt{2}\sqrt{2}\\ &= x-\sqrt{2x}-\sqrt{2x}+2=\boxed{x-2\sqrt{2x}+2}\end{aligned}$$

13. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

Solution: We work out x^2 first.

$$\begin{aligned}x^2 &= \left(2-\sqrt{3}\right)^2=\left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)\\ &= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3}\\ &= 4-4\sqrt{3}+3=7-4\sqrt{3}\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6.

$$\begin{aligned}x^2-4x+6 &= \left(2-\sqrt{3}\right)^2-4\left(2-\sqrt{3}\right)+6=\\ &= 7-4\sqrt{3}-8+4\sqrt{3}+6=7-8+6=\boxed{5}\end{aligned}$$

