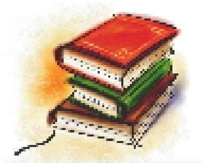


Sample Problems

1. Rationalize the denominator in each of the following expressions.

a) $\dfrac{3}{\sqrt{5}}$ b) $\dfrac{1}{\sqrt{10}-3}$ c) $\dfrac{2}{\sqrt{7}+1}$ d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$ e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

2. Simplify each of the following expressions.

a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$

b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)(5\sqrt{7}-\sqrt{3})$

c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$

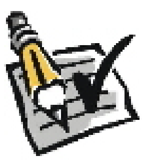
d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$

e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$

3. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2=4x+1$ b) $x^2+13=8x$



Practice Problems

1. Rationalize the denominator in each of the following expressions.

a) $\dfrac{4}{\sqrt{7}}$ c) $\dfrac{1}{\sqrt{10}+3}$ e) $\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$ g) $\dfrac{-5x+5}{\sqrt{x}+1}$

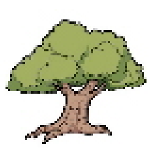
b) $\dfrac{1}{\sqrt{7}-3}$ d) $\dfrac{2}{\sqrt{17}+4}$ f) $\dfrac{\sqrt{10}+\sqrt{2}}{\sqrt{10}-\sqrt{2}}$ h) $\dfrac{2}{\sqrt{x}+4}$

2. Find the exact value of

a) x^2-6x+1 if $x=3-\sqrt{10}$ b) $b^2+8b-20$ if $b=\sqrt{5}-2$ c) $x^2-10x+16$ if $x=\sqrt{6}+5$

3. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2+47=14x$ b) $x^2+12x+31=0$ c) $x^2=2x+1$



Enrichment

Simplify each of the following expressions.

1. $\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}$ 2. $\dfrac{\sqrt{1008+\sqrt{2015}}-\sqrt{1008-\sqrt{2015}}}{\sqrt{2}}$

Answers

Sample Problems

1. a) $4\sqrt{2}$ b) $3\sqrt{5}$ c) $4x^2y\sqrt{3xy}$ d) $-4\sqrt{5}$ e) $\dfrac{2}{3}$ f) $13a^5\sqrt{5a}$ g) 3 h) $11-4\sqrt{7}$

   i) $-10+6\sqrt{3}$ j) $5x+\sqrt{5x}-6$ k) $6+\sqrt{x}-2x$ l) $x-2\sqrt{2x}+2$

2. a) $\dfrac{3\sqrt{5}}{5}$ b) $3+\sqrt{10}$ c) $\dfrac{-1+\sqrt{7}}{3}$ d) $3+\sqrt{5}$ e) $\dfrac{13-4\sqrt{10}}{3}$

3. a) 7 b) 172 c) 3 d) 4 e) 22 4. 5 5. a) $2-\sqrt{5},\ \ 2+\sqrt{5}$ b) $4-\sqrt{3},\ \ 4+\sqrt{3}$

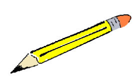
Practice Problems

1. a) $2\sqrt{5}$ b) $2\sqrt{3}$ c) $8\sqrt{5}$ d) $7a^2\sqrt{2a}$ e) 1 f) $9-4\sqrt{5}$ g) $17\sqrt{5}-38$ h) $103-\sqrt{5}$

2. a) $\dfrac{4\sqrt{7}}{7}$ b) $-\dfrac{3+\sqrt{7}}{2}$ c) $-3+\sqrt{10}$ d) $-8+2\sqrt{17}$ or $2\left(-4+\sqrt{17}\right)$ e) $\dfrac{3-\sqrt{5}}{2}$ f) $\dfrac{\sqrt{5}+3}{2}$

   g) $-5\sqrt{x}+5$ h) $\dfrac{2\sqrt{x}-8}{x-16}$ 3. a) 2 b) $-27+4\sqrt{5}$ c) -3

4. a) $7-\sqrt{2},\ \ 7+\sqrt{2}$ b) $-6-\sqrt{5},\ \ -6+\sqrt{5}$ c) $1-\sqrt{2},\ \ 1+\sqrt{2}$

Sample Problems  Solutions

1. Rationalize the denominator in each of the following expressions.

   a) $\dfrac{3}{\sqrt{5}}$

   Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\boxed{\frac{3\sqrt{5}}{5}}$$

   b) $\dfrac{1}{\sqrt{10}-3}$

   Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3}=\frac{1}{\sqrt{10}-3}\cdot\frac{\sqrt{10}+3}{\sqrt{10}+3}=\frac{\sqrt{10}+3}{1}=\boxed{3+\sqrt{10}}$$

   The denominator is 1 since

$$\left(\sqrt{10}-3\right)\left(\sqrt{10}+3\right)=\sqrt{10}\sqrt{10}+3\sqrt{10}-3\sqrt{10}-9=10-9=1$$

c) $\dfrac{2}{\sqrt{7}+1}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1}=\frac{2}{\sqrt{7}+1}\cdot\frac{\sqrt{7}-1}{\sqrt{7}-1}=\frac{2\left(\sqrt{7}-1\right)}{7-1}=\frac{2\left(\sqrt{7}-1\right)}{6}=\boxed{\frac{\sqrt{7}-1}{3}}$$

d) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$

Solution:

$$\frac{\sqrt{5}-1}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\left(\sqrt{5}-1\right)\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)}=\frac{5+2\sqrt{5}-\sqrt{5}-2}{5+2\sqrt{5}-2\sqrt{5}-4}=\frac{3+\sqrt{5}}{1}=\boxed{3+\sqrt{5}}$$

e) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

Solution: We multiply the fraction by 1 as a fraction of whose both numerator and denominator are the conjugate of the denominator.

$$\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot 1=\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}-\sqrt{5}}=\frac{\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}{\left(\sqrt{8}+\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}$$

We FOIL out both numerator and denominator

$$\frac{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}-\sqrt{5}\sqrt{8}+\sqrt{5}\sqrt{5}}{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}+\sqrt{5}\sqrt{8}-\sqrt{5}\sqrt{5}}=\frac{8-\sqrt{40}-\sqrt{40}+5}{8-5}=\frac{13-2\sqrt{40}}{3}$$

Note: although this answer is acceptable, the expression can be further simplified.

$$\frac{13-2\sqrt{40}}{3}=\frac{13-2\sqrt{4\cdot 10}}{3}=\frac{13-2\sqrt{4}\cdot\sqrt{10}}{3}=\frac{13-2\cdot 2\cdot\sqrt{10}}{3}=\boxed{\frac{13-4\sqrt{10}}{3}}$$

2. Simplify each of the following expressions.

a) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$

Solution:

$$\frac{3-\sqrt{5}}{3+\sqrt{5}}+\frac{3+\sqrt{5}}{3-\sqrt{5}}=\frac{\left(3-\sqrt{5}\right)^2+\left(3+\sqrt{5}\right)^2}{\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)}=\frac{14-6\sqrt{5}+14+6\sqrt{5}}{9-5}=\frac{28}{4}=\boxed{7}$$

b) $\left(\dfrac{8}{\sqrt{7}+\sqrt{3}}+\dfrac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)$

Solution: We first bring the fractions to the common denominator - they are conjugates, that helps, because the common denominator is then rational. We then multiply the factors.

$$\left(\frac{8}{\sqrt{7}+\sqrt{3}}+\frac{12}{\sqrt{7}-\sqrt{3}}\right)\left(5\sqrt{7}-\sqrt{3}\right)=$$

$$\begin{aligned}
&= \frac{8\left(\sqrt{7}-\sqrt{3}\right)+12\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)}\left(5\sqrt{7}-\sqrt{3}\right)\\
&= \frac{8\sqrt{7}-8\sqrt{3}+12\sqrt{7}+12\sqrt{3}}{7-3}\left(5\sqrt{7}-\sqrt{3}\right)\\
&= \frac{20\sqrt{7}+4\sqrt{3}}{4}\left(5\sqrt{7}-\sqrt{3}\right)=\left(5\sqrt{7}+\sqrt{3}\right)\left(5\sqrt{7}-\sqrt{3}\right)\\
&= 25\,(7)-3=175-3=\boxed{172}
\end{aligned}$$

c) $\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}}$
Solution:

$$\begin{aligned}\sqrt{\sqrt{41}+4\sqrt{2}}\cdot\sqrt{\sqrt{41}-\sqrt{32}} &= \sqrt{\sqrt{41}+\sqrt{32}}\cdot\sqrt{\sqrt{41}-\sqrt{32}} = \sqrt{\left(\sqrt{41}+\sqrt{32}\right)\left(\sqrt{41}-\sqrt{32}\right)} \\ &= \sqrt{\left(\left(\sqrt{41}\right)^2-\left(\sqrt{32}\right)^2\right)} = \sqrt{41-32} = \sqrt{9} = \boxed{3}\end{aligned}$$

d) $\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}}$
Solution:

$$\begin{aligned}\sqrt{5\sqrt{3}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}} &= \sqrt{\sqrt{75}+\sqrt{59}}\cdot\sqrt{\sqrt{75}-\sqrt{59}} = \sqrt{\left(\sqrt{75}+\sqrt{59}\right)\left(\sqrt{75}-\sqrt{59}\right)} \\ &= \sqrt{\left(\left(\sqrt{75}\right)^2-\left(\sqrt{59}\right)^2\right)} = \sqrt{75-59} = \sqrt{16} = \boxed{4}\end{aligned}$$

e) $\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2$
Solution:

$$\begin{aligned}\left(\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}\right)^2 &= \left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}} \\ &= 6+\sqrt{11}+6-\sqrt{11}+2\sqrt{\left(6+\sqrt{11}\right)\left(6-\sqrt{11}\right)} \\ &= 12+2\sqrt{36-11} = 12+2\sqrt{25} = 12+2\cdot 5 = \boxed{22}\end{aligned}$$

3. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.
Solution: We work out x^2 first.

$$\begin{aligned}x^2 &= \left(2-\sqrt{3}\right)^2 = \left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right) \\ &= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3} \\ &= 4-4\sqrt{3}+3 = 7-4\sqrt{3}\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6.

$$\begin{aligned}x^2-4x+6 &= \left(2-\sqrt{3}\right)^2-4\left(2-\sqrt{3}\right)+6 = \\ &= 7-4\sqrt{3}-8+4\sqrt{3}+6 = 7-8+6 = \boxed{5}\end{aligned}$$

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2=4x+1$
Solution: We factor by completing the square.

$$\begin{aligned}x^2 &= 4x+1 && \text{reduce one side to zero} \\ x^2-4x-1 &= 0 && (x-2)^2=x^2-4x+4 \\ \underbrace{x^2-4x+4}-4-1 &= 0 \\ (x-2)^2-5 &= 0 \\ (x-2)^2-\left(\sqrt{5}\right)^2 &= 0 \\ \left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right) &= 0\end{aligned}$$

$$\boxed{x_1 = 2-\sqrt{5} \quad \text{and} \quad x_2 = 2+\sqrt{5}}$$

We check: if $x = 2-\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2-\sqrt{5}\right)^2 = \left(2-\sqrt{5}\right)\left(2-\sqrt{5}\right) = 4-2\sqrt{5}-2\sqrt{5}+\sqrt{5}\sqrt{5} = 4-4\sqrt{5}+5 = 9-4\sqrt{5} \\ \text{RHS} &= 4\left(2-\sqrt{5}\right)+1 = 8-4\sqrt{5}+1 = 9-4\sqrt{5} \end{aligned}$$

and if $x = 2+\sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= \left(2+\sqrt{5}\right)^2 = \left(2+\sqrt{5}\right)\left(2+\sqrt{5}\right) = 4+2\sqrt{5}+2\sqrt{5}+\sqrt{5}\sqrt{5} = 4+4\sqrt{5}+5 = 9+4\sqrt{5} \\ \text{RHS} &= 4\left(2+\sqrt{5}\right)+1 = 8+4\sqrt{5}+1 = 9+4\sqrt{5} \end{aligned}$$

So, both solutions are correct.

b) $x^2 + 13 = 8x$

Solution: We factor by completing the square.

$$\begin{aligned} x^2-8x+13 &= 0 \qquad\qquad (x-4)^2 = x^2-8x+16 \\ \underbrace{x^2-8x+16}-16+13 &= 0 \\ (x-4)^2-3 &= 0 \\ (x-4)^2-\left(\sqrt{3}\right)^2 &= 0 \\ \left(x-4+\sqrt{3}\right)\left(x-4-\sqrt{3}\right) &= 0 \end{aligned}$$

$$\boxed{x_1 = 4-\sqrt{3} \ \text{ and } \ x_2 = 4+\sqrt{3}}$$

We check: if $x = 4-\sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= \left(4-\sqrt{3}\right)^2+13 = \left(4-\sqrt{3}\right)\left(4-\sqrt{3}\right)+13 = 16-4\sqrt{3}-4\sqrt{3}+3+13 = 32-8\sqrt{3} \\ \text{RHS} &= 8\left(4-\sqrt{3}\right) = 32-8\sqrt{3} \end{aligned}$$

and if $x = 4+\sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= \left(4+\sqrt{3}\right)^2+13 = \left(4+\sqrt{3}\right)\left(4+\sqrt{3}\right)+13 = 16+4\sqrt{3}+4\sqrt{3}+3+13 = 32+8\sqrt{3} \\ \text{RHS} &= 8\left(4+\sqrt{3}\right) = 32+8\sqrt{3} \end{aligned}$$

So, both solutions are correct.

