In this section, we will learn about some new set operations.

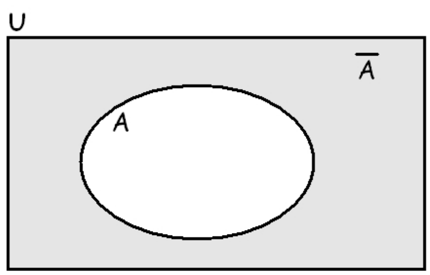
Definition: The set U usually denotes the universal set, i.e. the set everything that exists within the context of a problem.

If U is given as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then we will work out the problem as if only those 10 numbers exist. For that problem, this is all that exists. Until now we did not need the universal set. However, we will need it for our newly defined set operation.

Definition: Given a set A, the **complement of A**, (denoted by $\overline{A}$ or A' or A^C) is the set such that for all $x \in U$, x is an element of $\overline{A}$ if and only if x is not an element of A.

Example 1. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 5, 8, 10\}$, compute $\overline{A}$.

Solution: If $A = \{1, 2, 3, 5, 8, 10\}$, then $\overline{A} = \{4, 6, 7, 9\}$. The picture shows what region of the Venn diagram is the complement of A.



Example 2. If $U = \mathbb{N}$ and $E = \{2k : k \in \mathbb{N}\}$, find $\overline{E}$.

Solution: $E = \{2, 4, 6, 8, 10, 12, ...\}$ the set of all even natural numbers. To find $\overline{E}$, we need to list all natural numbers not in E. Therefore, $\overline{E} = \{1, 3, 5, 7, 9, 11, ...\}$.

Example 3. Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 4, 6, 10\}$, and $B = \{2, 3, 6, 7, 8, 10\}$. Find each of the following.

a) $\overline{A \cup B}$ b) $\overline{A} \cup \overline{B}$ c) $\overline{A \cap B}$ d) $\overline{A} \cap \overline{B}$

Solution: a) We first find $A \cup B$ and then take that set's complement.

$$A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\} \text{ and then } \overline{A \cup B} = \{5, 9\}$$

b) We first find the complement of A and B and then take the union of the two sets.

$$\begin{aligned} \overline{A} &= \{2, 3, 5, 7, 8, 9\} \text{ and } \overline{B} = \{1, 4, 9\} \\ \overline{A} \cup \overline{B} &= \{2, 3, 5, 7, 8, 9\} \cup \{1, 4, 9\} = \{1, 2, 3, 4, 5, 7, 8, 9\} \end{aligned}$$

c) We first find $A \cap B$ and then take that set's complement.

$$A \cap B = \{6, 10\} \text{ and then } \overline{A \cap B} = \{1, 2, 3, 4, 5, 7, 8, 9\}$$

d) We first find the complement of A and B and then take the intersection of the two sets.

$$\begin{aligned}\overline{A} &= \{2,3,5,7,8,9\} \text{ and } \overline{B} = \{1,4,5,9\} \\ \overline{A}\cap\overline{B} &= \{2,3,5,7,8,9\}\cap\{1,4,5,9\} = \{5,9\}\end{aligned}$$

Definition: Given a set A, the **cardinality of A**, (denoted by $|A|$ or by $n(A)$) is the number of elements in A.

Example 4. Suppose that $A = \{1,2,5,7\}$. Find $n(A)$.

Solution: We simply count the elements in A. Since A has four elements, $n(A) = 4$.

Example 5. List all subsets of the given set if

a) $A = \{1\}$ b) $B = \{1,2\}$ c) $C = \{1,2,3\}$ d) $D = \{1,2,3,4\}$

Solution: a) The set $\{1\}$ has two subsets: $\varnothing$ and $\{1\}$.

b) The set $\{1,2\}$ has four subsets: $\varnothing$, $\{1\}$, $\{2\}$, and $\{1,2\}$.

c) As the set given becomes larger, it becomes important to be systematic when listing the subsets. Let us list the subsets of C by organizing by their cardinalities.

0-element subsets:	$\varnothing$	
1-element subsets:	$\{1\}$, $\{2\}$, $\{3\}$	
2-element subsets:	$\{1,2\}$, $\{1,3\}$, $\{2,3\}$	
3-element subsets:	$\{1,2,3\}$	So there are 8 subsets.

d) We will list the subsets of D by organizing by their cardinalities.

0-element subsets:	$\varnothing$	1 subset	
1-element subsets:	$\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$	4 subsets	
2-element subsets:	$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,4\}$ $\{2,4\}$ $\{3,4\}$	6 subsets	
3-element subsets:	$\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$	4 subsets	So there are
4-element subsets:	$\{1,2,3,4\}$	1 subset	16 subsets.

The reader might have noticed that the number of subsets doubles when we add a new element to the set. This is indeed true.

This might sound strange at first, but we can also multiply sets.

Definition: Given non-empty sets A and B, the cartesian product of A and B, denoted by $A\times B$ is the set of all ordered pairs (a,b) where $a\in A$ and $b\in B$.

Example 6. Suppose that $P = \{1,2,3\}$ and $S = \{\mathrm{T},\mathrm{F}\}$. Find $P\times S$ and $S\times P$.

Solution: $P\times S$ is the set of all ordered pairs (p,s) where $p\in P$ and $s\in S$. This set is:

$$P\times S = \{(1,\mathrm{T}),(2,\mathrm{T}),(3,\mathrm{T}),(1,\mathrm{F}),(2,\mathrm{F}),(3,\mathrm{F})\}$$

On the other hand, the set $S \times P$ is the set of all ordered pairs (s, p) where $s \in S$ and $p \in P$. This set is:

$$S \times P = \{(\mathrm{T}, 1), (\mathrm{T}, 2), (\mathrm{T}, 3), (\mathrm{F}, 1), (\mathrm{F}, 2), (\mathrm{F}, 3)\}$$

Notice that this sort of multiplication is not commutative, i.e. $P \times S \neq S \times P$. We can also see why the name product: $n(A \times B) = n(A) \cdot n(B)$.

Practice Problems

1. This problem aims to show why we need the universal set when computing the complement of a set. Suppose that $A = \{1, 3, 4\}$
   a) Compute $\overline{A}$ if $U = \{1, 2, 3, 4, 5\}$.
   b) Compute $\overline{A}$ if $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
2. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 9, 10\}$, $B = \{2, 3, 4, 8, 10\}$, and $C = \{2, 7, 8, 10\}$. Find each of the following.
   a) $\overline{A} \cap \overline{B \cup C}$ b) $\overline{A} \cap \left(B \cup \overline{C}\right)$ c) $\left(\overline{A} \cap B\right) \cup \overline{C}$
3. Suppose that A is a set defined as $A = \{1, 2, 7\}$, $B = \{2, 4, 6, 8\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Determine whether the given statements are true or false.
   a) $A \cup \overline{A} = U$
   b) $A \cap \overline{A} = \varnothing$
   c) $\overline{A} \cap \overline{B} = \overline{A \cap B}$
   d) $n(A) + n\left(\overline{A}\right) = n(U)$
   e) $n(A \cup B) = n(A) + n(B)$
4. List all subsets of $A = \{1, 2, 3, 4, 5\}$
5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 3, 4, 6, 8\}$, and $C = \{1, 4, 5, 6, 7, 9\}$. Label each of the following statements as true or false.
   a) $A \subseteq (B \cup C)$ b) $9 \in \overline{A}$ c) $\overline{B} \cap \overline{C} = \overline{B \cup C}$
6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 3, 4, 6, 8\}$, and $C = \{1, 4, 5, 6, 7, 9\}$. Find each of the following.
   a) $A \cap B \cap C$
   b) $C \cap (B \cup A)$
   c) $B \cap \overline{C}$
   d) $\overline{A} \cup \overline{B}$
   e) $\overline{A \cap B}$
   f) $\left(B \cap \overline{C}\right) \cup C$
   g) $\left(A \cap \overline{B}\right) \cup \left(C \cap \overline{B}\right)$
   h) $n(A \cup C)$
7. List all two-element subsets of $C = \{1, 4, 5, 6, 7, 9\}$.
8. Suppose that $A = \{1, 2, 3, 4\}$ and $B = \{p, q\}$. Find $A \times B$ and $B \times A$.

Answers

1. a) $\{2, 5\}$ b) $\{2, 5, 6, 7, 8, 9, 10\}$

2. a) $\{5\}$ b) $\{3, 5, 8\}$ c) $\{1, 3, 4, 5, 6, 8, 9\}$

3. a) true b) true c) false d) true e) false

4. There are 32 subsets, listed below.

0-element subsets:	$\varnothing$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$	5 subsets
2-element subsets:	$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,4\}$ $\{2,4\}$ $\{3,4\}$ $\{1,5\}$ $\{2,5\}$ $\{3,5\}$ $\{4,5\}$	10 subsets
3-element subsets:	$\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$ $\{1,2,5\}$ $\{1,3,5\}$ $\{1,4,5\}$ $\{2,3,5\}$ $\{2,4,5\}$ $\{3,4,5\}$	10 subsets
4-element subsets:	$\{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}$	5 subsets
5-element subsets:	$\{1,2,3,4,5\}$	1 subset

5. a) true b) true c) true

6. a) $\{1\}$ b) $\{1, 4, 5, 6, 7\}$ c) $\{3, 8\}$ d) $\{2, 4, 5, 6, 7, 8, 9\}$ e) $\{2, 4, 5, 6, 7, 8, 9\}$
   f) $\{1, 3, 4, 5, 6, 7, 8, 9\}$ g) $\{5, 7, 9\}$ h) 7

7.
$\{1,4\}$
$\{1,5\}$ $\{4,5\}$
$\{1,6\}$ $\{4,6\}$ $\{5,6\}$
$\{1,7\}$ $\{4,7\}$ $\{5,7\}$ $\{6,7\}$
$\{1,9\}$ $\{4,9\}$ $\{5,9\}$ $\{6,9\}$ $\{7,9\}$

8. $A \times B = \{(1,p), (2,p), (3,p), (4,p), (1,q), (2,q), (3,q), (4,q)\}$ and
   $B \times A = \{(p,1), (p,2), (p,3), (p,4), (q,1), (q,2), (q,3), (q,4)\}$

