A very powerful proving technique is what we call **indirect proof**, or **proof by contradiction**.

The logic behind this proving technique is as follows. Suppose that we start with a true statement and arrive to other statements by making logically correct steps. Then these new statements must all be true.

Suppose we start with a statement and use logically correct steps to arrive to other statements, including one that is obviously false. Then we must have started with a false statement.

True statements only imply true statements. If our conclusion is false, we must have started with a false statement.

Suppose we want to prove a statement to be true. In case of a proof by contradiction, we formulate the exact opposite of our statement, and, using logically correct steps, we derive an obviously false statement. This proves that we started with a false statement. Therefore, the opposite of our statement is false, which means that our statement is true.

The fact that $\sqrt{2}$ is irrational can be proven by contradiction.

Definition: A number is **rational** if it can be written as a fraction of two integers.

Definition: A number is **irrational** if it is not rational, i.e. it can not be written as a fraction of two integers.

Theorem: $\sqrt{2}$ is an irrational number.

Proof. Suppose, for a contradiction, that $\sqrt{2}$ is rational, i.e. there exist two integers, a and b $(b \neq 0)$ such that

$$\sqrt{2} = \frac{a}{b}$$

We may also assume that the fraction $\frac{a}{b}$ is in lowest terms, otherwise we could reduce the fraction $\frac{a}{b}$ and replace it with the reduced equivalent. So, let us assume that $\frac{a}{b}$ is in lowest terms, which means that a and b do not share any divisor larger than 1. Now let us square both sides.

$$2 = \frac{a^2}{b^2}$$

Let us multiply both sides by b^2.

$$2b^2 = a^2$$

Since a^2 is twice another integer, it is even. This means that a itself must be even. Let us re-write $a = 2k$ where k is some integer.

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \end{aligned}$$

Let us divide both sides by 2. Then we have

$$b^2 = 2k^2$$

Since b^2 is twice another integer, it is even. This means that b itself must be even. We are now done, because the following statements cannot all be true.

1. a and b are two integers that do not share any divisors.
2. a is even.
3. b is even

This is a contradiciton, guaranteeing that there is at least one false statement among the three. This means that the assumption that $\sqrt{2}$ is rational must be false. This completes our proof. ■

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

