Systems become more complicated in several different ways. One way is that the equations are of higher degree than one. Quadratic systems will be studied later. Another way is that the equations remain linear, but the number of variables and equations grow. We will now learn how to solve systems in three variables. However, this method can be used with systems of equations with any number of unknowns.

If we have just one equation with two unknowns, we cannot solve for unique values of the unknowns. We can only solve for one unknown in terms of the other. Similarly, we can only completely solve a system in three variables if we have three equations.

The method we will learn is a very general method, a generalization of elimination. It is useful to keep in mind however, that symmetries in systems often enable us to solve it faster. It is worth to inspect a system for such shortcuts.

**Example 1.** Solve the given system of linear equations.

$$\begin{cases} x - y + z = -4 \\ -x + 3y - 2z = 11 \\ -x - 5y - z = 10 \end{cases}$$

**Solution:** We will not need new techniques if we just notice that we can solve for $y$ after we add the first and second equation.

$$\begin{cases} \quad x - \ y + z = -4 \\ + \ -x - 5y - z = 10 \end{cases}$$

$$-6y = 6 \quad \text{divide by } -6$$

$$y = -1$$

Now that we have the value of $y$, the problem is reduced to a system of linear equations in two variables that we already know how to handle.

$$\begin{cases} x - (-1) + z = -4 \\ -x + 3(-1) - 2z = 11 \\ -x - 5(-1) - z = 10 \end{cases} \implies \begin{cases} x + 1 + z = -4 \\ -x - 3 - 2z = 11 \\ -x + 5 - z = 10 \end{cases} \implies \begin{cases} x + z = -5 \\ -x - 2z = 14 \\ -x - z = 5 \end{cases}$$

The first equation and the third are dependent: if we multiply the first equation by $-1$, we get the third one. However, given that we have two unknown left, we only need two equations. So we discard the last equation and solve the remaining system.

$$\begin{cases} \quad x + z = -5 \\ + \ -x - 2z = 14 \end{cases}$$

$$-z = 9$$

$$z = -9$$

Now that we know the values of $y$ and $z$, we can easily find that of $x$ using either one of the equations.

$$x + z = -5 \implies x + (-9) = -5 \implies x - 9 = -5 \implies x = 4$$

So our solution is $\boxed{x = 4,\ y = -1, \text{ and } z = -9}$. We check:

$$\begin{cases} x - y + z = -4 \\ -x + 3y - 2z = 11 \\ -x - 5y - z = 10 \end{cases} \implies \begin{cases} \text{LHS} = 4 - (-1) + (-9) = 4 + 1 - 9 = 5 - 9 = -4 = \text{RHS} \ \checkmark \\ \text{LHS} = -4 + 3(-1) - 2(-9) = -4 - 3 + 18 = -7 + 18 = 11 = \text{RHS} \ \checkmark \\ \text{LHS} = -4 - 5(-1) - (-9) = -4 + 5 + 9 = 1 + 9 = 10 = \text{RHS} \ \checkmark \end{cases}$$

So our solution is correct.

The following example does not offer such shortcuts, so we will use a general method which is a generalization of elimination. The computations are all easy, the difficulty lies in keeping track of the equations. We will number the equations to keep track of them.

**Example 2.** Solve the given system of linear equations.

$$\begin{cases} 1) \quad 3x - y + 5z = 9 \\ 2) \quad 2x + 3y + z = 29 \\ 3) \quad -4x - 2y - 3z = -36 \end{cases}$$

**Solution:** First, we inspect the three equations and look for an unknown with the simplest coefficient, $1$ and $-1$ highly preferred. In this case, $y$ in the first equation and $z$ in the second equation are the best candidates. We will choose $y$ in the first equation.

We will use 'up' the first equation to eliminate $y$ from the second and third equations.

To eliminate $y$ from the second equation, we multiply both sides of the first equation by three and then add the two equations. The resulting equation will be our new second equation.

$$\begin{array}{ll} 1) & 3x - y + 5z = 9 \quad / \cdot 3 \\ 2) & 2x + 3y + z = 29 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} 1) & 9x - 3y + 15z = 27 \\ 2) & 2x + 3y + z = 29 \end{array}$$

We add the two equations:

$$\begin{array}{ll} 1) & 9x - 3y + 15z = 27 \\ 2) & + \quad 2x + 3y + z = 29 \\ \hline & 11x + 16z = 56 \end{array}$$

This is our new second equation.

We repeat the process. This time we use the first equation to eliminate $y$ from the third equation. The resulting equation will be our new third equation. To eliminate $y$, we multiply the first equation by $-2$ and then add the two equations.

$$\begin{array}{ll} 1) & 3x - y + 5z = 9 \quad / \cdot (-2) \\ 3) & -4x - 2y - 3z = -36 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} 1) & -6x + 2y - 10z = -18 \\ 3) & -4x - 2y - 3z = -36 \end{array}$$

We add the two equations:

$$\begin{array}{ll} 1) & -6x + 2y - 10z = -18 \\ 3) & + \quad -4x - 2y - 3z = -36 \\ \hline & -10x - 13z = -54 \end{array}$$

This is our new third equation.

We now obtained a system of equations with two variables:

$$\begin{array}{ll} 2) & -10x - 13z = -54 \\ 3) & 11x + 16z = 56 \end{array}$$

We already know how to solve those. We will use elimination. We will multiply the first equation by 11 and the second equation by 10 and then we add the two equations.

$$\begin{array}{ll} 2) & -10x - 13z = -54 \quad / \cdot 11 \\ 3) & 11x + 16z = 56 \quad / \cdot 10 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} 2) & -110x - 143z = -594 \\ 3) & 110x + 160z = 560 \end{array}$$

We add the two equations:

$$\begin{array}{ll} 2) & -110x - 143z = -594 \\ 3) & + \quad 110x + 160z = 560 \\ \hline & 17z = -34 \quad \text{divide by 17} \\ & z = -2 \end{array}$$

Now we can solve for $x$ in the second equation:

2) $-10x-13z=-54$ becomes

$$\begin{aligned} -10x-13(-2) &= -54 && \text{Solve for } x\\ -10x+26 &= -54 && \text{subtract } 26\\ -10x &= -80 && \text{divide by } -10\\ x &= 8 \end{aligned}$$

Now that we know the values of $x$ and $z$, we can easily solve for $y$ in either of the equations. We will use the first equation.

1) $3x-y+5z=9$ becomes

$$\begin{aligned} 3\cdot 8-y+5(-2) &= 9 && \text{Solve for } y\\ 24-y-10 &= 9 && \text{combine like terms}\\ 14-y &= 9 && \text{subtract } 14\\ -y &= -5 && \text{multiply by } -1\\ y &= 5 \end{aligned}$$

Our solution is: $\boxed{x=8,\ y=5, \text{ and } z=-2}$. We check:

1) $\text{LHS}=3x-y+5z=3\cdot 8-5+5(-2)=24-5-10=9=\text{RHS}$ ✓

2) $\text{LHS}=2x+3y+z=2\cdot 8+3\cdot 5+(-2)=16+15-2=29=\text{RHS}$ ✓

3) $\text{LHS}=-4x-2y-3z=-4\cdot 8-2\cdot 5-3(-2)=-32-10+6=-36\ \text{RHS}$ ✓

Therefore, our solution is correct.

The following example is an application of systems with three equations.

**Example 3.** Find the equation of a parabola if we know that its graph contains the points $(-1,15)$, $(2,6)$, and $(4,10)$.

**Solution:** We start by setting up the coefficients in the formula as the unknown. Suppose that $y=ax^2+bx+c$ is the equation. Finding the equation of the parabole is equivalent to finding the values of $a$, $b$, and $c$. To find these three unknowns, we will need three equations. For that, we will use the three points given on the graph of the parabola.

$(-1,15)$ is on the graph $\Longrightarrow a(-1)^2+b(-1)+c=15$

$(2,6)$ is on the graph $\Longrightarrow a\cdot 2^2+b\cdot 2+c=6$

$(4,10)$ is on the graph $\Longrightarrow a\cdot 4^2+b\cdot 4+c=10$

We simplify each of the equations:
$$\begin{cases} 1)\quad a-b+c=15\\ 2)\quad 4a+2b+c=6\\ 3)\quad 16a+4b+c=10 \end{cases}$$

We will first use the first equation to eliminate $c$ from the second and third equations.

$$\begin{aligned} &1)\quad a-b+c=15 \quad /\cdot(-1) & & & 1)\quad -a+b-c=-15\\ &2)\quad 4a+2b+c=6 & \Longrightarrow & & 2)\quad 4a+2b+c=6 \end{aligned}$$

We add the two equations:

$$\begin{aligned} 1)\quad & -a+b-c=-15\\ 2)\quad + & \ \ 4a+2b+c=6\\ \hline & 3a+3b=-9 \quad \text{divide by } 3\\ & a+b=-3 \end{aligned}$$

Similarly, we will use the first equation to eliminate $c$ from the third equation.

$$\begin{array}{ll} 1) & a - b + c = 15 \quad / \cdot (-1) \\ 3) & 16a + 4b + c = 10 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} 1) & -a + b - c = -15 \\ 3) & 16a + 4b + c = 10 \end{array}$$

We add the two equations:

$$\begin{array}{lrl} 1) & -a + b - c &= -15 \\ 3) & + \quad 16a + 4b + c &= 10 \\ \hline & 15a + 5b &= -5 \quad \text{divide by } 5 \\ & 3a + b &= -1 \end{array}$$

We have reduced the problem to a system in $a$ and $b$ only. We solve the system.

$$\begin{array}{ll} 2) & a + b = -3 \quad / \cdot (-1) \\ 3) & 3a + b = -1 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} 2) & -a - b = 3 \\ 3) & 3a + b = -1 \end{array}$$

We add the two equations:

$$\begin{array}{lrl} 2) & -a - b &= 3 \\ 3) & + \quad 3a + b &= -1 \\ \hline & 2a &= 2 \quad \text{divide by } 2 \\ & a &= 1 \end{array}$$

Then the second equation, $-a - b = 3$ becomes $-1 - b = 3 \implies b = -4$

Then the first equation, $a - b + c = 15$ becomes $1 - (-4) + c = 15 \implies c = 10$

So $a = 1$, $b = -4$, and $c = 10$, and this means that the parabola is $\boxed{y = x^2 - 4x + 10}$.

We check:

If $x = -1$, then $y = (-1)^2 - 4(-1) + 10 = 1 + 5 + 10 = 15$, so $(-1, 15)$ is on the graph. ✓

If $x = 2$, then $y = 2^2 - 4 \cdot 2 + 10 = 4 - 8 + 10 = 6$, so $(2, 6)$ is on the graph. ✓

If $x = 4$, then $y = 4^2 - 4 \cdot 4 + 10 = 16 - 16 + 10 = 10$, so $(4, 10)$ is also on the graph. ✓

## Practice Problems

1. Solve each of the following system of equations.

a) $\begin{cases} 2x - y + z = -1 \\ x + 2y + 2z = 5 \\ -3x - 4y - 3z = 2 \end{cases}$ c) $\begin{cases} -5x + 3y - 2z = 44 \\ 2x - y + 3z = -31 \\ -x + 4y + 7z = -9 \end{cases}$ e) $\begin{cases} 2x + 3y + 7z = 3 \\ 4x - 5y - 3z = 19 \\ 5x - 3y + z = 21 \end{cases}$

b) $\begin{cases} -2a - 3b + c = 5 \\ 3a - 2b + c = -6 \\ 6a + 4b + 3c = 31 \end{cases}$ d) $\begin{cases} 4a - 3b + 5c = -3 \\ -2a + 5b - 3c = 11 \\ -a + 2b - 4c = -1 \end{cases}$ f) $\begin{cases} 2a - 5b - c = 7 \\ -3a + 10b + c = -7 \\ a - 8b + 3c = 15 \end{cases}$

2. Find the equation of the parabola if we know that the points $(1, 4)$, $(-2, 10)$, and $(3, -10)$ are all points on the graph of the parabola.

3. Find the equation of the parabola if we know that the points $(2, 1)$, $(-8, 11)$, and $(4, -13)$ are all points on the graph of the parabola.

4. We went to the market to buy some fruit. We decided to buy some apples, strawberries, and oranges. If we buy 2 apples, 3 boxes of strawberry, and 4 oranges, the fruit would cost \$15.30. If we buy 1 box of strawberry, 4 apples, and 2 oranges, the fruit would cost \$10.90. If we buy 1 orange, 5 apples and 2 boxes of strawberry, the fruit would cost \$13.70. What is the price of each type of fruit?

## Answers

1. a) Solve each of the following system of equations.

a) $x = -5, y = -2, z = 7$ b) $a = -3, b = 4, c = 11$ c) $x = 0, y = 10, z = -7$

d) $a = -1, b = 3, c = 2$ e) $x = 8, y = 5, z = -4$ f) $a = 15, b = 3, c = 8$

2. $y = -x^2 - 3x + 8$ 3. $y = -\frac{1}{2}x^2 - 4x + 11$ 4. apple: \$1.50 strawberry: \$2.50 orange: \$1.20



 