Part 1 - When the Leading Coefficient is 1

Part 1 - Sample Problems

Factor each of the following by completing the square.

1.) $-4x + x^2 - 21$ 2.) $32 - 18x + x^2$ 3.) $28x + x^2 - 1173$ 4.) $17 - 2a + a^2$

Part 1- Practice Problems

Factor each of the following by completing the square.

1. $x^2 - 10x + 21$
2. $x^2 - 6x + 8$
3. $22y + y^2 + 105$
4. $b^2 - 4b - 45$
5. $14a + a^2 - 51$
6. $b^2 - 10b + 26$
7. $3 + x^2 - 4x$
8. $d^2 + 2d + 2$
9. $6x + x^2 - 432$
10. $x^2 - 14x + 58$
11. $m^2 - 42m + 432$
12. $x^2 - 50x + 525$
13. $10y + y^2 - 375$
14. $x^2 - 40x + 336$
15. $x^2 - 6x + 25$
16. $q^2 - 2q - 48$
17. $x^2 - 18x + 81$
18. $t^2 - 36t - 4437$
19. $x^2 - 46x + 360$
20. $14q + q^2 - 2352$

Part 1 - Sample Problems - Answers

1.) $(x+3)(x-7)$ 2.) $(x-2)(x-16)$ 3.) $(x+51)(x-23)$ 4.) does not factor

Part 1 - Practice Problems - Answers

1.) $(x-3)(x-7)$
2.) $(x-2)(x-4)$
3.) $(y+15)(y+7)$
4.) $(b+5)(b-9)$
5.) $(a+17)(a-3)$
6.) does not factor
7.) $(x-1)(x-3)$
8.) does not factor
9.) $(x+24)(x-18)$
10.) does not factor
11.) $(m-18)(m-24)$
12.) $(x-15)(x-35)$
13.) $(y+25)(y-15)$
14.) $(x-12)(x-28)$
15.) does not factor
16.) $(q+6)(q-8)$
17.) $(x-9)^2$
18.) $(t+51)(t-87)$
19.) $(x-10)(x-36)$
20.) $(q+56)(q-42)$

Part 1 - Sample Problems - Solutions

Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". Recall that a **complete square** or a **perfect square** is the square of a sum or a difference. For example, the expressions $(2a-3)^2$ and $(x+7)^2$ are complete (or perfect) squares.

1. Factor $-4x+x^2-21$ by completing the square.

Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x+x^2-21=x^2-4x-21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying x, **sign included**. In our example, the magic number is $\dfrac{-4}{2}=-2$. We do not write this line in the main computation.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x-2)^2=(x-2)(x-2)=x^2-2x-2x+4=x^2-4x+4$$

Step 2C. We write the "helper line" $(x-2)^2=x^2-4x+4$ in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x+x^2-21=x^2-4x-21 \qquad\qquad (x-2)^2=x^2-4x\ \boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, x^2-4x as our expression to be factored. We can see that the last term, $+4$ is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2-4x\qquad-21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2-4x\ {\color{blue}+4-4}-21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$\begin{aligned} x^2-4x-21 &= \underbrace{x^2-4x+4}-4-21 \\ &= (x-2)^2-25 \end{aligned}$$

Step 5. We re-write the last number as a square.

$$(x-2)^2-25=(x-2)^2-5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2-5^2=(x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5) = (x+3)(x-7)$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result, $(x+3)(x-7)$ is correct.

The entire computation should look like this:

$$-4x + x^2 - 21 =$$

$$\begin{aligned} &= x^2 - 4x - 21 \qquad\qquad (x-2)^2 = x^2 - 4x + 4 \\ &= \underbrace{x^2 - 4x + 4}-4 - 21 \\ &= (x-2)^2 - 25 \\ &= (x-2)^2 - 5^2 \\ &= (x-2+5)(x-2-5) \\ &= (x+3)(x-7) \end{aligned}$$

We check:

$$(x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$$

2. Factor $32 - 18x + x^2$ by completing the square.

Step 1. We re-arrange the terms by decreasing order of degree.

$$32 - 18x + x^2 = x^2 - 18x + 32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying x, sign included.) In our example, the magic number is $\frac{-18}{2} = -9$.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x-9)^2 = (x-9)(x-9) = x^2 - 9x - 9x + 81 = x^2 - 18x + 81$$

Step 2C. We write the "helper line" $(x-9)^2 = x^2 - 18x + 81$ in the upper right hand side of the paper.

$$x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x + \boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, $x^2 - 18x$ as our expression to be factored. We can see that the last term, $+81$ is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2 - 18x + 32 = x^2 - 18x + 81 - 81 + 32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$\begin{aligned} x^2 - 18x + 32 &= \underbrace{x^2 - 18x + 81} - 81 + 32 \\ &= (x-9)^2 - 49 \end{aligned}$$

Step 5. We re-write the last number as a square.

$$(x-9)^2 - 49 = (x-9)^2 - 7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-9)^2 - 7^2 = (x-9+7)(x-9-7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-9+7)(x-9-7) = (x-2)(x-16)$$

Step 8. We check back by multiplication. (See below.)

The entire computation should look like this:

$$\begin{aligned} 32 - 18x + x^2 &= \\ &= x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x + 81 \\ &= \underbrace{x^2 - 18x + 81} - 81 + 32 \\ &= (x-9)^2 - 49 \\ &= (x-9)^2 - 7^2 \\ &= (x-9+7)(x-9-7) \\ &= (x-2)(x-16) \end{aligned}$$

We check:

$$(x-2)(x-16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$$

Thus our result, $(x-2)(x-16)$ is correct.

3. Factor $28x + x^2 - 1173$ by completing the square.

$$\begin{aligned} 28x + x^2 - 1173 &= \qquad \text{rearrange terms} \\ x^2 + 28x - 1173 &= \qquad \text{the "magic number" is } \frac{28}{2} = 14 \end{aligned}$$

We work out $(x+14)^2 = x^2 + 28x + 196$ on the margin.

$$\begin{aligned} x^2 + 28x - 1173 &= \qquad (x+14)^2 = x^2 + 28x \boxed{+196} \text{ so we smuggle in 196} \\ \underbrace{x^2 + 28x + 196} - 196 - 1173 &= \qquad \text{realize comple square, combine like terms} \\ (x+14)^2 - 1369 &= \qquad \text{from calculator, } \sqrt{1369} = 37 \\ (x+14)^2 - 37^2 &= \qquad \text{difference of squares theorem} \\ (x+14+37)(x+14-37) &= \qquad \text{combine like terms} \\ &= (x+51)(x-23) \end{aligned}$$

We check by multiplication:

$$(x+51)(x-23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$$

Thus our result,$(x+51)(x-23)$ is correct.

4. Factor $17 - 2a + a^2$ by completing the square.

   Solution:

$$\begin{aligned} 17 - 2a + a^2 &= \quad \text{rearrange terms} \\ a^2 - 2a + 17 &= \quad \text{the "magic number" is } \frac{-2}{2} = -1 \end{aligned}$$

We work out $(a-1)^2 = a^2 - 2a + 1$ on the margin.

$$\begin{aligned} a^2 - 2a + 17 &= \quad (a-1)^2 = a^2 - 2a \boxed{+1}\text{, so we smuggle in 1} \\ \underbrace{a^2 - 2a + 1} - 1 + 17 &= \quad \text{realize comple square, combine like terms} \\ &= (a-1)^2 + 16 \end{aligned}$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares does not factor**, and so the expression $a^2 - 2a + 17$ can not be factored.

Part 2 - When the Leading Cofficient is Not 1 but can be factored out

Part 2 - Sample Problems

Factor each of the following by completing the square.

1. $18x - 3x^2 + 165$
2. $267x^2 - 48x^3 + 3x^4$
3. $5x^2 - 240x + 2160$
4. $9 - y^2 - 8y$

Part 2 - Practice Problems

Completely factor each of the following by completing the square.

1. $4x + 2x^2 - 30$
2. $70a^2 - 255a + 5a^3$
3. $78b^2 - 30b^3 + 3b^4$
4. $32x + 2x^2 - 594$
5. $18c - 24c^2 + 6c^3$
6. $-2d - 2d^2 - d^3$
7. $432 - x^2 - 6x$
8. $x^2 - 14x + 58$
9. $10abc - 600ac + 5ab^2c$
10. $70y^3 + 24y^4 + 2y^5$
11. $18x^2y^2 - 216x^2y + 3x^2y^3$
12. $1000x - 50x^2 - 5x^3$

Part 2 - Sample Problems - Answers

1.) $-3(x+5)(x-11)$ 2.) $3x^2(x^2-16x+89)$ 3.) $5(x-12)(x-36)$ 4.) $-(y+9)(y-1)$

Part 2 - Practice Problems - Answers

1.) $2(x+5)(x-3)$ 2.) $5a(a-3)(a+17)$ 3.) $3b^2(b^2-10b+26)$ 4.) $2(x+27)(x-11)$

5.) $6c(c-1)(c-3)$ 6.) $-d(d^2+2d+2)$ 7.) $-(x+24)(x-18)$ 8.) does not factor

9.) $5ac(b+12)(b-10)$ 10.) $2y^3(y+7)(y+5)$ 11.) $3x^2y(y+12)(y-6)$ 12.) $-5x(x+20)(x-10)$

Part 2 - Sample Problems - Solutions

1. Factor $18x - 3x^2 + 165$ by completing the square.

   Step 1. We re-arrange the terms by decreasing order of degree.

   $$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

   Step 2. We factor out the greatest common factor.

   $$-3x^2 + 18x + 165 = -3\left(x^2 - 6x - 55\right)$$

   Step 3. We factor the expression within the parentheses by completing the square.

   $$\begin{aligned} -3\left(x^2 - 6x - 55\right) &= \qquad\qquad (x-3)^2 = x^2 - 6x + 9 \\ -3\left(\underbrace{x^2 - 6x + 9}{} - 9 - 55\right) &= \\ -3\left((x-3)^2 - 64\right) &= \\ -3\left((x-3)^2 - 8^2\right) &= \\ -3(x-3+8)(x-3-8) &= -3(x+5)(x-11) \end{aligned}$$

   Step 4. We check our result by multiplication.

   $$-3(x+5)(x-11) = -3\left(x^2 - 11x + 5x - 55\right) = -3\left(x^2 - 6x - 55\right) = -3x^2 + 18x + 165$$

   Thus our result, $-3(x+5)(x-11)$ is correct.

2. Factor $267x^2 - 48x^3 + 3x^4$ by completing the square.

   $$\begin{aligned} 267x^2 - 48x^3 + 3x^4 &= \qquad \text{rearrange terms} \\ 3x^4 - 48x^3 + 267x^2 &= \qquad \text{factor out } 3x^2 \\ 3x^2\left(x^2 - 16x + 89\right) &= \qquad (x-8)^2 = x^2 - 16x \boxed{+64} \\ 3x^2\left(\underbrace{x^2 - 16x + 64}{} - 64 + 89\right) &= \qquad \text{realize comple square, combine like terms} \\ &= 3x^2\left((x-8)^2 + 25\right) \end{aligned}$$

   We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares does not factor. Thus the expression $3x^2\left(x^2 - 16x + 89\right)$ is completely factored.

3. Factor $5x^2 - 240x + 2160$ by completing the square.

$$5x^2 - 240x + 2160 = \qquad \text{factor out 5}$$

$$\begin{aligned}
&= 5\left(x^2 - 48x + 432\right) && (x-24)^2 = x^2 - 48x \boxed{+576} \\
&= 5\left(\underbrace{x^2 - 48x + 576} - 576 + 432\right) \\
&= 5\left((x-24)^2 - 144\right) \\
&= 5\left((x-24)^2 - 12^2\right) \\
&= 5(x-24+12)(x-24-12) \\
&= 5(x-12)(x-36)
\end{aligned}$$

Check:

$$5(x-12)(x-36) = 5\left(x^2 - 12x - 36x + 432\right) = 5\left(x^2 - 48x + 432\right) = 5x^2 - 240x + 2160$$

Thus our result, $5(x-12)(x-36)$ is correct.

4. Factor $9 - y^2 - 8y$ by completing the square.

$$9 - y^2 - 8y = \qquad \text{rearrange terms}$$

$$\begin{aligned}
-y^2 - 8y + 9 &= && \text{factor out } -1 \\
-1\left(y^2 + 8y - 9\right) &= && (y+4)^2 = y^2 + 8y \boxed{+16} \\
-\left(\underbrace{y^2 + 8y + 16} - 16 - 9\right) &= && \text{realize comple square, combine like terms} \\
-\left((y+4)^2 - 25\right) &= && \text{re-write 25 as a square} \\
-\left((y+4)^2 - 5^2\right) &= && \text{factor via the difference of squares theorem} \\
-\left((y+4+5)(y+4-5)\right) &= && \text{combine like terms, drop extra parentheses} \\
&= -(y+9)(y-1)
\end{aligned}$$

We check:

$$-(y+9)(y-1) = -\left(y^2 - y + 9y - 9\right) = -\left(y^2 + 8y - 9\right) = -y^2 - 8y + 9$$

Thus our result, $-(y+9)(y-1)$ is correct.

Part 3 - With Fractions

It is strongly recommended to use fractions and not decimals. The steps are identical as before (see Part 1 and Part 2), the only complications are that we need to perform the same computations with fractions.

Part 3 - Sample Problems

Completely factor each of the following by completing the square.

1. $54x - 6x^2 + 60$
2. $5p + p^2 + 6$
3. $3x^2 - 4x - 319$
4. $11x + 6x^2 - 10$
5. $a^2 + ab + b^2$

Part 3 - Practice Problems

Completely factor each of the following by completing the square.

1. $x + x^2 - 12$
2. $x^2 - x - 90$
3. $11x + x^2 + 30$
4. $x^2 - 17x + 72$
5. $3x + 3x^2 - 60$
6. $2x^2 - 6x + 17$
7. $x + 2x^2 - 1$
8. $112x + 2x^2 - 2x^3$
9. $5a^2 - 14a - 3$
10. $33c^4 - 270c^3 - c^5$
11. $m + 6m^2 - 2$
12. $6x^2 - 7x - 3$
13. $10p^2 - 11p - 6$
14. $15x^2 - 34x + 15$

Part 3 - Sample Problems- Answers

1. $-6(x+1)(x-10)$
2. $(p+3)(p+2)$
3. $3\left(x+\frac{29}{3}\right)(x-11)=(3x+29)(x-11)$
4. $6\left(x+\frac{5}{2}\right)\left(x-\frac{2}{3}\right)=(2x+5)(3x-2)$
5. does not factor

Part 3 - Practice Problems - Answers

1. $(x+4)(x-3)$
2. $(x+9)(x-10)$
3. $(x+6)(x+5)$
4. $(x-8)(x-9)$
5. $3(x+5)(x-4)$
6. does not factor
7. $2(x+1)\left(x-\frac{1}{2}\right)=(x+1)(2x-1)$
8. $-2x(x+7)(x-8)$
9. $5\left(a+\frac{1}{5}\right)(a-3)=(5a+1)(a-3)$
10. $-c^3(c-15)(c-18)$
11. $6\left(m+\frac{2}{3}\right)\left(m-\frac{1}{2}\right)=(3m+2)(2m-1)$
12. $6\left(x+\frac{1}{3}\right)\left(x-\frac{3}{2}\right)=(3x+1)(2x-3)$
13. $10\left(p+\frac{2}{5}\right)\left(p-\frac{3}{2}\right)=(5p+2)(2p-3)$
14. $15\left(x-\frac{3}{5}\right)\left(x-\frac{5}{3}\right)=(5x-3)(3x-5)$

Part 3 - Sample Problems - Solutions

It is strongly recommended to use fractions and not decimals. The steps are identical as before (see Part 1 and Part 2), the only complications are that we need to perform the same computations with fractions.

1. Factor $54x - 6x^2 + 60$ by completing the square.

$$\begin{aligned} 54x - 6x^2 + 60 &= && \text{rearrange terms} \\ -6x^2 + 54x + 60 &= && \text{factor out } -6 \\ -6\left(x^2 - 9x - 10\right) &= && \text{the "magic number" is } \frac{-9}{2} = -\frac{9}{2} \end{aligned}$$

We work out $\left(x - \frac{9}{2}\right)^2$ on the margin:

$$\begin{aligned} \left(x - \frac{9}{2}\right)^2 &= \left(x - \frac{9}{2}\right)\left(x - \frac{9}{2}\right) = x^2 - \frac{9}{2}x - \frac{9}{2}x + \frac{81}{4} \\ &= x^2 - \frac{18}{2}x + \frac{81}{4} = x^2 - 9x + \frac{81}{4} \end{aligned}$$

$$\begin{aligned} -6\left(x^2 - 9x - 10\right) &= && \left(x - \frac{9}{2}\right)^2 = x^2 - 9x + \frac{81}{4} \text{ so we smuggle in } \frac{81}{4} \\ -6\left(\underbrace{x^2 - 9x + \frac{81}{4}} - \frac{81}{4} - 10\right) &= && \text{realize comple square, re-write 10 as } \frac{40}{4} \\ -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{40}{4}\right) &= && \text{combine like terms} \\ -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{121}{4}\right) &= && \text{re-write } \frac{121}{4} \text{ as } \left(\frac{11}{2}\right)^2 \\ -6\left(\left(x - \frac{9}{2}\right)^2 - \left(\frac{11}{2}\right)^2\right) &= && \text{factor via the difference of squares theorem} \\ -6\left(x - \frac{9}{2} + \frac{11}{2}\right)\left(x - \frac{9}{2} - \frac{11}{2}\right) &= && \text{combine like terms} \\ -6\left(x + \frac{2}{2}\right)\left(x - \frac{20}{2}\right) &= && \text{simplify fractions} \\ &= -6(x+1)(x-10) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} -6(x+1)(x-10) &= -6\left(x^2 - 10x + x - 10\right) = -6\left(x^2 - 9x - 10\right) \\ &= -6x^2 + 54x + 60 \end{aligned}$$

Thus our result, $-6(x+1)(x-10)$ is correct.

2. Factor $5p+p^2+6$ by completing the square.

$$\begin{aligned} 5p+p^2+6 &= \quad \text{rearrange terms} \\ p^2+5p+6 &= \quad \text{there is no GCF} \end{aligned}$$

The magic number is $\dfrac{5}{2}$. We work out $\left(p+\dfrac{5}{2}\right)^2$ on the margin:

$$\begin{aligned} \left(p+\frac{5}{2}\right)^2 &= \left(p+\frac{5}{2}\right)\left(p+\frac{5}{2}\right) = p^2+\frac{5}{2}p+\frac{5}{2}p+\frac{25}{4} \\ &= p^2+\frac{10}{2}p+\frac{25}{4} = p^2+5p+\frac{25}{4} \end{aligned}$$

$$\begin{aligned} p^2+5p+6 &= \quad \left(p+\frac{5}{2}\right)^2 = p^2+5p+\frac{25}{4} \quad \text{so we smuggle in } \frac{25}{4} \\ \underbrace{p^2+5p+\frac{25}{4}}-\frac{25}{4}+6 &= \quad \text{realize comple square, re-write 6 as } \frac{24}{4} \\ \left(p+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{24}{4} &= \quad \text{combine like terms} \\ \left(p+\frac{5}{2}\right)^2-\frac{1}{4} &= \quad \text{re-write } \frac{1}{4} \text{ as } \left(\frac{1}{2}\right)^2 \\ \left(p+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2 &= \quad \text{factor via the difference of squares theorem} \\ \left(p+\frac{5}{2}+\frac{1}{2}\right)\left(p+\frac{5}{2}-\frac{1}{2}\right) &= \quad \text{combine like terms} \\ \left(p+\frac{6}{2}\right)\left(p+\frac{4}{2}\right) &= \quad \text{simplify fractions} \\ &= (p+3)(p+2) \end{aligned}$$

We check by multiplication:

$$(p+3)(p+2) = p^2+2p+3p+6 = p^2+5p+6$$

Thus our result, $(p+3)(p+2)$ is correct.

Note: These polynomials can easily be factored using trial and error and other methods. Why should we use completing the square? All other methods will break down once the numbers are less friendly. Then ONLY completing the square will work. Since those computations will be more difficult, you should learn this method while numbers are easy.

3. Factor $3x^2 - 4x - 319$ by completing the square.

If the leading coefficient is not 1, we will factor it out before completing the square.

$$3x^2 - 4x - 319 = 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right)$$

Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\left(x - \frac{2}{3}\right)^2 = \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} = x^2 - \frac{4}{3}x + \frac{4}{9}$$

Thus we smuggle in $\frac{4}{9}$. The computation:

$$\begin{aligned}
3x^2 - 4x - 319 &= \\
&= 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) && \left(x - \frac{2}{3}\right)^2 = x^2 - \frac{4}{3}x + \frac{4}{9} \\
&= 3\left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}} - \frac{4}{9} - \frac{319}{3}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{319}{3}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{319 \cdot 3}{3 \cdot 3}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{961}{9}\right) && \frac{961}{9} = \left(\frac{31}{3}\right)^2 \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2\right) \\
&= 3\left(x - \frac{2}{3} + \frac{31}{3}\right)\left(x - \frac{2}{3} - \frac{31}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)\left(x - \frac{33}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)(x - 11) = (3x + 29)(x - 11)
\end{aligned}$$

We check:

$$(3x + 29)(x - 11) = 3x^2 - 33x + 29x - 319 = 3x^2 - 4x - 319$$

4. Factor $11x + 6x^2 - 10$ by completing the square.

We first factor out the leading coefficient.

$$11x + 6x^2 - 10 = 6x^2 + 11x - 10 = 6\left(x^2 + \frac{11}{6}x - \frac{5}{3}\right)$$

Half of the linear coefficient is $\frac{11}{6} \div 2 = \frac{11}{6}\left(\frac{1}{2}\right) = \frac{11}{12}$ and so we FOIL $\left(x + \frac{11}{12}\right)^2$.

$$\begin{aligned}\left(x + \frac{11}{12}\right)^2 &= \left(x + \frac{11}{12}\right)\left(x + \frac{11}{12}\right) \\ &= x^2 + \frac{11}{12}x + \frac{11}{12}x + \left(\frac{11}{12}\right)^2 & \qquad \frac{11}{12} + \frac{11}{12} = \frac{22}{12} = \frac{11}{6} \\ &= x^2 + \frac{11}{6}x + \frac{121}{144} & \qquad \text{thus we will smuggle in } \frac{121}{144}\end{aligned}$$

$$\begin{aligned}6\left(x^2 + \frac{11}{6}x - \frac{5}{3}\right) &= \\ 6\left(\underbrace{x^2 + \frac{11}{6}x + \frac{121}{144}} - \frac{121}{144} - \frac{5}{3}\right) &= & \qquad \frac{5}{3} = \frac{5 \cdot 48}{3 \cdot 48} = \frac{240}{144} \\ 6\left(\left(x + \frac{11}{12}\right)^2 - \frac{121}{144} - \frac{240}{144}\right) &= \\ 6\left(\left(x + \frac{11}{12}\right)^2 - \frac{361}{144}\right) &= & \qquad \sqrt{361} = 19 \quad \text{and} \quad \sqrt{144} = 12 \\ 6\left(\left(x + \frac{11}{12}\right)^2 - \left(\frac{19}{12}\right)^2\right) &= \\ 6\left(x + \frac{11}{12} + \frac{19}{12}\right)\left(x + \frac{11}{12} - \frac{19}{12}\right) &= \\ 6\left(x + \frac{30}{12}\right)\left(x - \frac{8}{12}\right) &= \\ 6\left(x + \frac{5}{2}\right)\left(x - \frac{2}{3}\right) &= \\ 2\left(x + \frac{5}{2}\right)3\left(x - \frac{2}{3}\right) &= (2x+5)(3x-2)\end{aligned}$$

We check:

$$(2x+5)(3x-2) = 6x^2 - 4x + 15x - 10 = 6x^2 + 11x - 10$$

Thus our answer, $(2x+5)(3x-2)$ is correct.

5. a^2+ab+b^2

Solution: This expression comes up when we factor the difference of cubes theorem. It is easy to verify that $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$. Completing the square enables us to prove that the second, longer factor is irreducible, i.e. can not be further factored. Consider a to be our variable, and b like the numbers inthe previous example. Then a^2 is quadratic, ab is linear, and b^2 is the constant term. We proceed to complete the square. Half of the linear coefficient is $\frac{1}{2}b$. So the complete square we need is $\left(a+\frac{b}{2}\right)^2$

$$\left(a+\frac{b}{2}\right)^2=a^2+\frac{ab}{2}+\frac{ab}{2}+\frac{b^2}{4}=a^2+ab+\frac{b^2}{4}$$

So we will smuggle in $\frac{b^2}{4}$

$$\begin{aligned} a^2+ab+b^2 &= \\ a^2+ab+\frac{b^2}{4}-\frac{b^2}{4}+b^2 &= \\ \left(a+\frac{b}{2}\right)^2-\frac{b^2}{4}+\frac{4b^2}{4} &= \\ \left(a+\frac{b}{2}\right)^2+\frac{3}{4}b^2 &= \end{aligned}$$

Since the sum of two squares can not be factored, the expression a^2+ab+b^2 is irreducible, can not be factored.

