Prove each of the following statements by induction.

1. For all natural numbers n,

   a) $1+2+3+...+n=\dfrac{n(n+1)}{2}$.

   b) $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

   c) $1^3+2^3+3^3+...+n^3=\dfrac{n^2(n+1)^2}{4}$.

2. For all natural numbers n, $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+...+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

3. For all natural numbers n, $1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+....+n\cdot 2^n=2(1+(n-1)2^n)$.

4. For all natural numbers $n\geq 5$, $2^n\geq n^2$.

5. Prove that for all natural number n, 10^n can be written as a sum of two perfect squares.

6. Recall (F_n) is the Fibonacci sequence, defined as

$$F_1=1, \quad F_2=1, \quad \text{and} \quad F_{n+2}=F_n+F_{n+1}$$

   Prove each of the following statements for all natural numbers n.

   a) $F_1+F_2+F_3+...+F_n=F_{n+2}-1$

   b) $F_1^2+F_2^2+F_3^2+...+F_n^2=F_nF_{n+1}$

   c) $F_1+F_3+...+F_{2n-1}=F_{2n}$

   d) $F_2+F_4+...+F_{2n}=F_{2n+1}-1$

   e) For all $n\geq 2$, $F_n^2-F_{n-1}\cdot F_{n+1}=(-1)^{n-1}$

7. Prove that for all natural number n, 11^n-6 is divisible by 5.

8. Prove that for all natural number n, $2^{3n}-3^n$ is divisible by 5.

9. Prove that for all natural number n, $2^{2n}+24n-10$ is divisible by 18.

Sample Problems - Solutions

1. a) For all natural numbers n, $1+2+3+...+n=\dfrac{n(n+1)}{2}$.

Proof:
Part 1: We check for the first few values of n.

If $n=1$, then

$$\text{LHS}=1=1 \qquad \text{and} \quad \text{RHS}=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

If $n=2$, then

$$\text{LHS}=1+2=3 \qquad \text{and} \quad \text{RHS}=\frac{2(2+1)}{2}=\frac{6}{2}=3$$

If $n=3$, then

$$\text{LHS}=1+2+3=6 \qquad \text{and} \quad \text{RHS}=\frac{3(3+1)}{2}=\frac{12}{2}=6$$

So the statement is true for $n=1,2$ and 3.

Part 2. Suppose that k is a positive integer for which

$$1+2+3+...+k=\frac{k(k+1)}{2} \qquad \text{This is the inductional hypotheses}$$

Let us add $k+1$ to both sides. The left-hand side becomes

$$1+2+3+...+k+(k+1)$$

and the right hand side becomes

$$\begin{aligned}\text{RHS} &= \frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2}\\ &= \frac{k^2+k+2k+2}{2}=\frac{k^2+3k+2}{2}=\frac{(k+1)(k+2)}{2}=\frac{(k+1)((k+1)+1)}{2}\end{aligned}$$

So we have proved that for all positive integers k, if

$$1+2+3+...+k=\frac{k(k+1)}{2} \quad \text{is true}$$

then

$$1+2+3+...+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$

which completes our proof.

b) For all natural numbers n, $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Proof:
Part 1: We check for the first few values of n.

If $n=1$, then

$$\text{LHS}=1^2=1 \quad \text{and} \quad \text{RHS}=\frac{1(1+1)(2\cdot 1+1)}{6}=\frac{6}{6}=1$$

If $n=2$, then

$$\text{LHS}=1^2+2^2=5 \quad \text{and} \quad \text{RHS}=\frac{2(2+1)(2\cdot 2+1)}{6}=\frac{30}{6}=5$$

If $n=3$, then

$$\text{LHS}=1^2+2^2+3^2=14 \quad \text{and} \quad \text{RHS}=\frac{3(3+1)(2\cdot 3+1)}{6}=\frac{84}{6}=14$$

So the statement is true for $n=1,2$ and 3.

Part 2. Suppose that k is a positive integer for which

$$1^2+2^2+3^2+...+k^2=\frac{k(k+1)(2k+1)}{6} \qquad \text{This is the inductional hypotheses}$$

Let us add $(k+1)^2$ to both sides. The left-hand side becomes

$$1^2+2^2+3^2+...+k^2+(k+1)^2$$

and the right hand side becomes

$$\text{RHS}=\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}$$

We factor out $\dfrac{k+1}{6}$

$$\begin{aligned}\text{RHS} &= \frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}=\frac{k+1}{6}(k(2k+1)+6(k+1))\\ &= \frac{k+1}{6}\left(2k^2+k+6k+6\right)=\frac{k+1}{6}\left(2k^2+7k+6\right)\end{aligned}$$

and so happens $2k^2+7k+6$ factors as $(k+2)(2k+3)$. So we have

$$\begin{aligned}\text{RHS} &= \frac{k+1}{6}\left(2k^2+7k+6\right)=\frac{k+1}{6}(k+2)(2k+3)=\frac{(k+1)(k+2)(2k+3)}{6}\\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\end{aligned}$$

So we have proved that for all positive integers k, if

$$1^2+2^2+3^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$$

then

$$1^2+2^2+3^2+...+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

which completes our proof.

c) For all natural numbers n, $1^3+2^3+3^3+...+n^3=\dfrac{n^2(n+1)^2}{4}$

Proof:

Part 1: We check for the first few values of n.

If $n=1$, then

$$\text{LHS}=1^3=1 \qquad \text{and } \ \text{RHS}=\frac{1^2(1+1)^2}{4}=\frac{4}{4}=1$$

If $n=2$, then

$$\text{LHS}=1^3+2^3=9 \qquad \text{and } \ \text{RHS}=\frac{2^2(2+1)^2}{4}=\frac{36}{4}=9$$

If $n=3$, then

$$\text{LHS}=1^3+2^3+3^3=36 \qquad \text{and } \ \text{RHS}=\frac{3^2(3+1)^2}{2^2}=\frac{144}{4}=36$$

So the statement is true for $n=1,2$ and 3.

Part 2. Suppose that k is a positive integer for which

$$1^3+2^3+3^3+...+k^3=\frac{k^2(k+1)^2}{4} \qquad \text{This is the inductional hypotheses}$$

Let us add $(k+1)^3$ to both sides. The left-hand side becomes

$$1^3+2^3+3^3+...+k^3+(k+1)^3$$

and the right hand side becomes

$$\begin{aligned}\text{RHS} &= \frac{k^2(k+1)^2}{4}+(k+1)^3=\frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \qquad \text{factor out } \frac{(k+1)^2}{4}\\ &= \frac{(k+1)^2}{4}\left(k^2+4(k+1)\right)=\frac{(k+1)^2}{4}\left(k^2+4k+4\right)=\frac{(k+1)^2}{4}(k+2)^2=\frac{(k+1)^2(k+1+1)^2}{4}\end{aligned}$$

So we have proved that for all positive integers k, if

$$1^3+2^3+3^3+...+k^3=\frac{k^2(k+1)^2}{4} \ \text{ is true}$$

then

$$1^3+2^3+3^3+...+k^3+(k+1)^3=\frac{(k+1)^2((k+1)+1)^2}{4}$$

which completes our proof.

2. For all natural numbers n, $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+...+\dfrac{1}{n\,(n+1)}=\dfrac{n}{n+1}$.

   Proof: Induction on n.
   Part 1. If $n=1$, then
$$\text{LHS}=\frac{1}{1\cdot 2}=\frac{1}{2}\quad\text{and}\quad\text{RHS}=\frac{1}{1+1}=\frac{1}{2}$$
   If $n=2$, then
$$\text{LHS}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}\quad\text{and}\quad\text{RHS}=\frac{2}{2+1}=\frac{2}{3}$$
   Part 2. Suppose that k is a natural number such that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{k\,(k+1)}=\frac{k}{k+1}\qquad\text{(Induction hypothesis)}$$
   We will add $\dfrac{1}{(k+1)\,(k+1+1)}=\dfrac{1}{(k+1)\,(k+2)}$ to both sides.
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{k\,(k+1)}+\frac{1}{(k+1)\,(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)\,(k+2)}$$
   Let us simplify the right hand side
$$\begin{aligned}\frac{k}{k+1}+\frac{1}{(k+1)\,(k+2)} &= \frac{k\,(k+2)}{(k+1)\,(k+2)}+\frac{1}{(k+1)\,(k+2)}=\frac{k\,(k+2)+1}{(k+1)\,(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)\,(k+2)}=\frac{(k+1)^2}{(k+1)\,(k+2)}=\frac{k+1}{k+2}\end{aligned}$$
   Thus we have that from the induction hypotheses
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{k\,(k+1)}=\frac{k}{k+1}$$
   the statement
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{(k+1)\,(k+2)}=\frac{k+1}{k+2}$$
   follows. This completes our proof.

3. For all natural numbers n, $1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+....+n\cdot 2^n=2\,(1+(n-1)\,2^n)$.

   Proof:

   Part 1. If $n=1$, then
$$\text{LHS}=1\cdot 2=2\quad\text{and}\quad\text{RHS}=2\left(1+(1-1)\,2^1\right)=2\cdot 1=2$$
   If $n=2$, then
$$\text{LHS}=1\cdot 2+2\cdot 2^2=10\quad\text{and}\quad\text{RHS}=2\left(1+(2-1)\,2^2\right)=2\cdot 5=10$$
   Part 2. Suppose that k is a natural number such that
$$1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+....+k\cdot 2^k=2\left(1+(k-1)\,2^k\right)\qquad\text{(Induction hypothesis)}$$
   Let us add $(k+1)\,2^{k+1}$ to both sides. The left-hand side becomes
$$\text{LHS}=1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+....+k\cdot 2^k+(k+1)\,2^{k+1}$$

and the right-hand side becomes

$$\begin{aligned}
\text{RHS} &= 2\left(1+(k-1)\,2^k\right)+(k+1)\,2^{k+1} = 2\left(1+(k-1)\,2^k\right)+(k+1)\,2^k\cdot 2 \qquad \text{factor out 2}\\
&= 2\left(1+(k-1)\,2^k+(k+1)\,2^k\right) = 2\left(1+k2^k-2^k+k2^k+2^k\right) =\\
&= 2\left(1+2k2^k\right) = 2\left(1+k2^{k+1}\right)
\end{aligned}$$

which completes our proof.

4. For all natural numbers $n \geq 5$, $2^n \geq n^2$

Note: this is a very interesting example illustrating how induction works, how we need both parts to work together to form a logically sound proof. The statement seems to be true immediately at $n = 1$ but then surprisingly, it will be false for a few values of n.

Proof:

Part 1. If $n = 1$, then

$$\text{LHS} = 2^1 = 2 \quad \text{and} \quad \text{RHS} = 1^2 = 1 \text{ and } \ 2 > 1 \quad \text{is true}$$

If $n = 2$, then

$$\text{LHS} = 2^2 = 4 \quad \text{and} \quad \text{RHS} = 2^2 = 4 \text{ and } \ 4 > 4 \ \text{ is false!}$$

If $n = 3$, then

$$\text{LHS} = 2^3 = 8 \quad \text{and} \quad \text{RHS} = 3^2 = 9 \text{ and } \ 8 > 9 \ \text{ is false!}$$

If $n = 4$, then

$$\text{LHS} = 2^4 = 16 \quad \text{and} \quad \text{RHS} = 4^2 = 16 \text{ and } \ 16 > 16 \ \text{ is false!}$$

If $n = 5$, then

$$\text{LHS} = 2^5 = 32 \quad \text{and} \quad \text{RHS} = 5^2 = 25 \text{ and } \ 32 > 25 \ \text{ is true}$$

If $n = 6$, then

$$\text{LHS} = 2^6 = 64 \quad \text{and} \quad \text{RHS} = 6^2 = 36 \text{ and } \ 64 > 36 \ \text{ is true}$$

At this point we have a sense that the statement will stay true because the left-hand side doubles when we go from k to $k+1$ while the right-hand side just increases from k^2 to $(k+1)^2$. The proof in part 2 will formalize this idea, that between the two types of growth, doubling is much 'faster'.

All we need to show is that moving from k^2 to $(k+1)^2$ is a smaller increment than doubling. In short, that

$$\begin{aligned}
(k+1)^2 &< 2k^2 \\
k^2+2k+1 &< 2k^2 \qquad \text{subtract } k^2\\
2k+1 &< k^2
\end{aligned}$$

This should be easy to prove for most positive integers. We can either solve the quadratic inequality or be a little bit sloppy or generous and say that if k is greater than 1, then

$$\begin{aligned}
1 &< k \qquad \text{add } 2k\\
2k+1 &< 2k+k\\
2k+1 &< 3k
\end{aligned}$$

and if k is greater than 3, then

$$\begin{aligned}
3 &< k \quad \text{multiply by } k > 0\\
3k &< k^2
\end{aligned}$$

In short, if $k > 3$, then $2k+1 < 3k < k^2$ and so

$$\begin{aligned} 2k+1 &< k^2 \qquad \text{add } k^2 \text{ to both sides} \\ k^2+2k+1 &< 2k^2 \\ (k+1)^2 &< 2k^2 \end{aligned}$$

This will be the core of the proof in part 2.

Part 2. Suppose that k is a natural number such that

$$2^k > k^2 \qquad \text{(Induction hypothesis)}$$

Let us multiply both sides by 2. The left-hand side becomes

$$\text{LHS} = 2 \cdot 2^k = 2^{k+1}$$

and the right-hand side becomes

$$\text{RHS} = 2k^2$$

So we have

$$\begin{aligned} 2^k &> k^2 \\ 2^{k+1} &> 2k^2 \end{aligned}$$

If $k > 3$, then

$$2^{k+1} > 2k^2 = k^2 + k^2 > k^2 + 3k > k^2 + 2k + 1 = (k+1)^2$$

and so we have that if $2^k > k^2$ is true AND $k > 3$, then $2^{k+1} > (k+1)^2$.

So we could easily prove the 'inheritance' property for all integers greater than 3 but the statement itself is NOT true for $n = 3$. Both components of the proof work together starting at $n = 5$.

	statement	inheritance to the next number
$n=1$	true	false
$n=2$	false	false
$n=3$	false	true (but we didn't prove it)
$n=4$	false	true
$n=5$	true	true

This table illustrates why induction only works here for $n \geq 5$.

5. Prove that for all natural number n, 10^n can be written as a sum of two perfect squares.

Proof: Part 1:

$n=1:$ $10^1 = 10 = 9 + 1 = 3^2 + 1^2$ ✓

$n=2:$ $10^2 = 100 = 64 + 36 = 8^2 + 6^2$ ✓

$n=3:$ $10^3 = 1000 = 900 + 100 = 30^2 + 90^2$ ✓

$n=4:$ $10^4 = 10\,000 = 6400 + 3600 = 80^2 + 60^2$ ✓ =

This problem illustrates that it is worth looking at the first part beyond $n = 1$ because it often helps us discover how the second part goes. In this case, the 'inheritance property' is from n to $n+2$.

Part 2. Suppose that k is a natural number such that $10^k = a^2 + b^2$ for some integers a and b. Consider now 10^{k+2}.

$$10^{k+2} = 10^k \cdot 10^2 = 100 \cdot 10^k = 100\left(a^2+b^2\right) = 100a^2 + 100b^2 = (10a)^2 + (10b)^2$$

So, if the statement is true for n, then it is also true for $n+2$.

$1 \Rightarrow 3 \Rightarrow 5 \Rightarrow 7 \Rightarrow 9 \Rightarrow 11 \Rightarrow 13....$ and $2 \Rightarrow 4 \Rightarrow 6 \Rightarrow 8 \Rightarrow 10 \Rightarrow 12....$

This completes our proof.

6. Recall (F_n) is the Fibonacci sequence, defined as $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_n + F_{n+1}$$

Prove each of the following statements for all natural numbers n.

a) $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$

Proof: Let us first label the terms of the sequence.

1	1	2	3	5	8	13	21	34	55	89
F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}

Part 1. $n = 1: \quad F_1 = F_3 - 1 \iff 1 = 2 - 1 \quad 1 = 1\checkmark$

$n = 2: \quad F_1 + F_2 = F_4 - 1 \iff 1 + 1 = 3 - 1 \quad 2 = 2\checkmark$

$n = 3: \quad F_1 + F_2 + F_3 = F_6 - 1 \iff 1 + 1 + 2 = 5 - 1 \quad 4 = 4\checkmark$

$n = 4: \quad F_1 + F_2 + F_3 + F_4 = F_8 - 1 \iff 1 + 1 + 2 + 3 = 8 - 1 \quad 7 = 7\checkmark$

Part 2. IH. Suppose that k is a natural number such that $F_1 + F_2 + F_3 + ... + F_k = F_{k+2} - 1$. To move to the next integer, we add F_{k+1} to both sides.

$$\begin{aligned}
F_1 + F_2 + F_3 + ... + F_k &= F_{k+2} - 1 && \text{add } F_{k+1} \\
F_1 + F_2 + F_3 + ... + F_k + F_{k+1} &= F_{k+2} - 1 + F_{k+1} && F_{k+1} + F_{k+2} = F_{k+3} \\
F_1 + F_2 + F_3 + ... + F_k + F_{k+1} &= F_{k+3} - 1 \\
F_1 + F_2 + F_3 + ... + F_k + F_{k+1} &= F_{(k+1)+2} - 1
\end{aligned}$$

Therefore, if the statement is true for k, then it is also true for $k + 1$. This completes our proof.

b) $F_1^2 + F_2^2 + F_3^2 + ... + F_n^2 = F_n F_{n+1}$

Proof:

Part 1. $n = 1: \quad F_1^2 = 1^2 = 1$ and $F_1 F_2 = 1 \cdot 1 \quad 1 = 1\checkmark$

$n = 2: \quad F_1^2 + F_2^2 = 1^2 + 1^2 = 2$ and $F_2 F_3 = 1 \cdot 2 \quad 2 = 2\checkmark$

$n = 3: \quad F_1^2 + F_2^2 + F_3^2 = 1^2 + 1^2 + 2^2 = 6$ and $F_3 F_4 = 2 \cdot 3 = 6 \quad 6 = 6\checkmark$

$n = 4: \quad F_1^2 + F_2^2 + F_3^2 + F_4^2 = 1^2 + 1^2 + 2^2 + 3^2 = 15$ and $F_4 F_5 = 3 \cdot 5 = 15 \quad 15 = 15\checkmark$

Part 2. IH. Suppose that k is a natural number such that $F_1^2 + F_2^2 + F_3^2 + ... + F_k^2 = F_k F_{k+1}$. To move to the next integer, we add F_{k+1}^2 to both sides.

$$\begin{aligned}
F_1^2 + F_2^2 + F_3^2 + ... + F_k^2 &= F_k F_{k+1} && \text{add } F_{k+1}^2 \\
F_1^2 + F_2^2 + F_3^2 + ... + F_k^2 + F_{k+1}^2 &= F_k F_{k+1} + F_{k+1}^2 \\
&= F_{k+1}(F_k + F_{k+1}) && F_k + F_{k+1} = F_{k+2} \\
&= F_{k+1} F_{k+2} = F_{k+1} F_{(k+1)+1}
\end{aligned}$$

Therefore, if the statement is true for k, then it is also true for $k + 1$. This completes our proof.

c) $F_1 + F_3 + ... + F_{2n-1} = F_{2n}$

Part 1. $n = 1:$ $F_1 = 1$ and $F_2 = 1$ $1 = 1\checkmark$

$n = 2:$ $F_1 + F_3 = 1 + 2 = 3$ and $F_4 = 3$ $3 = 3\checkmark$

$n = 3:$ $F_1 + F_3 + F_5 = 1 + 2 + 5 = 8$ and $F_6 = 8$ $8 = 8\checkmark$

$n = 4:$ $F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21$ and $F_8 = 21$ $21 = 21\checkmark$

Part 2. IH. Suppose that k is a natural number such that $F_1 + F_3 + ... + F_{2k-1} = F_{2k}$. To move to the next integer, we add F_{2k+1} to both sides.

$$\begin{aligned} F_1 + F_3 + ... + F_{2k-1} &= F_{2k} && \text{add } F_{2k+1} \\ F_1 + F_3 + ... + F_{2k-1} + F_{2k+1} &= F_{2k} + F_{2k+1} = F_{2k+2} \end{aligned}$$

This is the same as

$$F_1 + F_3 + ... + F_{2k-1} + F_{2(k+1)-1} = F_{2(k+1)}$$

Therefore, if the statement is true for k, then it is also true for $k + 1$. This completes our proof.

d) $F_2 + F_4 + ... + F_{2n} = F_{2n+1} - 1$

Part 1. $n = 1$ $F_2 = 1$ and $F_3 - 1 = 2 - 1$ $1 = 1\checkmark$

$n = 2$ $F_2 + F_4 = 1 + 3 = 4$ and $F_5 - 1 = 5 - 1$ $3 = 3\checkmark$

$n = 3$ $F_2 + F_4 + F_6 = 1 + 3 + 8 = 12$ and $F_7 - 1 = 13 - 1$ $12 = 12\checkmark$

$n = 4$ $F_2 + F_4 + F_6 + F_8 = 1 + 3 + 8 + 21 = 33$ and $F_9 - 1 = 34 - 1$ $33 = 33\checkmark$

Part 2. IH. Suppose that k is a natural number such that $F_2 + F_4 + ... + F_{2k} = F_{2k+1} - 1$. To move to the next integer, we add F_{2k+2} to both sides.

$$\begin{aligned} F_2 + F_4 + ... + F_{2k} &= F_{2k+1} - 1 && \text{add } F_{2k+2} \\ F_2 + F_4 + ... + F_{2k} + F_{2k+2} &= F_{2k+1} - 1 + F_{2k+2} \\ &= F_{2k+1} + F_{2k+2} - 1 && F_{2k+1} + F_{2k+2} = F_{2k+3} \\ &= F_{2k+3} - 1 \end{aligned}$$

This is the same as $F_2 + F_4 + ... + F_{2k} + F_{2(k+1)} = F_{2(k+1)+1} - 1$

Therefore, if the statement is true for k, then it is also true for $k + 1$. This completes our proof.

e) For all $n \geq 2$, $F_n^2 - F_{n-1} \cdot F_{n+1} = (-1)^{n-1}$

Part 1. $n = 2:$ $F_2^2 - F_1 \cdot F_3 = 1^2 - 1 \cdot 2 = 1 - 2 = -1$ and $(-1)^1 = -1$ $-1 = -1\checkmark$

$n = 3:$ $F_3^2 - F_2 \cdot F_4 = 2^2 - 1 \cdot 3 = 4 - 3 = 1$ and $(-1)^2 = 1$ $1 = 1\checkmark$

$n = 4:$ $F_4^2 - F_3 \cdot F_5 = 3^2 - 2 \cdot 5 = 9 - 10 = -1$ and $(-1)^3 = -1$ $-1 = -1\checkmark$

$n = 5:$ $F_5^2 - F_4 \cdot F_6 = 5^2 - 3 \cdot 8 = 25 - 24 = 1$ and $(-1)^4 = 1$ $1 = 1\checkmark$

Part 2. IH. The right-hand side is 1 for $n = 1, 3, 5, 7, ...$ and -1 for $n = 2, 4, 6, 8, ...$ For this reason, we would prove the inheritance property if we could show that the leftt-hand sides in case of k and $k + 1$ are opposites of each other.

$$F_{k+1}^2 - F_k F_{k+2} = -\left(F_k^2 - F_{k-1} F_{k+1}\right)$$

Let's try. We start with $F_{k+1}^2 - F_k F_{k+2}$. We will re-write F_{k+2} as $F_k + F_{k+1}$

$$\begin{aligned} F_{k+1}^2 - F_k F_{k+2} &= F_{k+1}^2 - F_k \left(F_k + F_{k+1}\right) && \text{distribute } F_k \\ &= F_{k+1}^2 - F_k^2 - F_k F_{k+1} && \text{swap terms} \\ &= -F_k^2 + F_{k+1}^2 - F_k F_{k+1} && \text{factor out } F_{k+1} \\ &= -F_k^2 + F_{k+1} \left(F_{k+1} - F_k\right) && F_{k+1} - F_k = F_{k-1} \\ &= -F_k^2 + F_{k+1} F_{k-1} = -\left(F_k^2 - F_{k-1} F_{k+1}\right) \end{aligned}$$

Therefore, if the statement is true for k, then it is also true for $k + 1$. This completes our proof.

7. Prove that for all natural number n, $11^n - 6$ is divisible by 5.

   Proof:

   Part 1. $n = 1$ $\quad 11^1 - 6 = 11 - 6 = 5$ ✓

   $n = 2$ $\quad 11^2 - 6 = 121 - 6 = 115$ ✓

   Part 2: IH. Suppose that k is a natural number such that $11^k - 6$ is divisible by 5. Then there exists integer α such that $11^k - 6 = 5\alpha$. Consider now $11^{k+1} - 6$

   $$11^{k+1} - 6 = 11 \cdot 11^k - 6 = (10+1)\,11^k - 6 = 10 \cdot 11^k + 11^k - 6 = 5\left(2 \cdot 11^k\right) + 5\alpha = 5\left(2 \cdot 11^k + \alpha\right)$$

   Therefore, if $11^k - 6$ is divisible by 5, then so is $11^{k+1} - 6$. This completes our proof.

8. Prove that for all natural number n, $2^{3n} - 3^n$ is divisible by 5.

   Proof: Part 1:

   $n = 1:$ $\quad 2^3 - 3^1 = 8 - 3 = 5$ ✓

   $n = 2:$ $\quad 2^6 - 3^2 = 64 - 9 = 55$ ✓

   Part 2: IH. Suppose that k is a natural number such that $2^{3k} - 3^k$ is divisible by 5. Then there exists integer α such that $2^{3k} - 3^k = 5\alpha$. Consider now $2^{3(k+1)} - 3^{k+1}$

   $$\begin{aligned} 2^{3(k+1)} - 3^{k+1} &= 2^{3k+3} - 3^{k+1} = 2^{3k} \cdot 2^3 - 3^k \cdot 3 = 8 \cdot 2^{3k} - 3 \cdot 3^k = (5+3) \cdot 2^{3k} - 3 \cdot 3^k \\ &= 5 \cdot 2^{3k} + 3 \cdot 2^{3k} - 3 \cdot 3^k = 5 \cdot 2^{3k} + 3\left(2^{3k} - 3^k\right) = 5 \cdot 2^{3k} + 3\,(5\alpha) = 5\left(2^{3k} + 3\alpha\right) \end{aligned}$$

   Therefore, if $2^{3k} - 3^k$ is divisible by 5, then so is $2^{3(k+1)} - 3^{k+1}$. This completes our proof.

9. Prove that for all natural number n, $2^{2n} + 24n - 10$ is divisible by 18.

   Proof: Part 1:

   $n = 1:$ $\quad 2^2 + 24 - 10 = 28 - 10 = 18$ ✓

   $n = 2:$ $\quad 2^4 + 24 \cdot 2 - 10 = 16 + 48 - 10 = 64 - 10 = 54 = 18 \cdot 3$ ✓

   Part 2: IH. Suppose that k is a natural number such that $2^{2k} + 24k - 10$ is divisible by 18. Then there exists integer α such that $2^{2k} + 24k - 10 = 18\alpha$. Consider now $2^{2(k+1)} + 24\,(k+1) - 10$

   $$\begin{aligned} 2^{2(k+1)} + 24\,(k+1) - 10 &= 2^{2k+2} + 24k + 24 - 10 = 2^{2k} \cdot 2^2 + 24k + 14 = 4 \cdot 2^{2k} + 24k + 3 \cdot 24k - 3 \cdot 24k + 14 \\ &= 4 \cdot 2^{2k} + 4 \cdot 24k - 40 + 40 - 72k + 14 = 4\left(2^{2k} + 24k - 10\right) - 72k + 54 \\ &= 4 \cdot 18\alpha - 18\,(4k - 3) = 18\,(4\alpha - 4k + 3) \end{aligned}$$

   Therefore, if $2^{2k} + 24k - 10$ is divisible by 18, then so is $2^{2(k+1)} + 24\,(k+1) - 10$. This completes our proof.

