Definition: **The symbol $\log_a b$ represents the number that if we write as an exponent of a, we achieve b.** This expression is only meaningful if both a and b are positive numbers and $a \neq 1$.

Every logarithmic statement can be re-written as an exponential statement.

$$\log_a b = x \text{ is the same as } a^x = b$$

Example: Find the value of $\log_2 8$.
$\log_2 8$ is a number, it is the exponent we need to write over 2 to obtain 8. In short, $2^? = 8$. The answer is clearly 3, since $2^3 = 8$.

$$\log_2 8 = 3$$

Similarly, most exponential statements can be re-written as a logarithmic statement.

$$n^y = M \text{ is the same as } \log_n M = y$$

Some properties of logarithms follow immediately from its definition.

Theorem 1. $\log_a a^k = k$

Proof. To achieve a^k, we need to raise a exactly to the kth power. Or, using exponential notation,

$$\text{if } x = \log_a a^k, \quad \text{then } a^x = a^k \quad \Longrightarrow \quad x = k$$

Sample Problems

1. Find the exact value of each of the following logarithmic expressions.

   a) $\log_3 3$ b) $\log_4 1$ c) $\log_5 \left(\frac{1}{25}\right)$ d) $\log_9 3$ e) $\log_8 \left(\frac{1}{16}\right)$ f) $\ln e^8$

2. Simplify each of the following expressions. Assume that $a > 0$ and $a \neq 1$.

   a) $\log_a \left(a^9\right)$ b) $\log_2 \left(8^x\right)$ c) $\log_2 \left(16^a\right)$ d) $\log_{25} \left(5^{36}\right)$ e) $\log_{25} \left(5^x\right)$ f) $\ln \left(\frac{1}{\sqrt{e}}\right)$ g) $\log_{10} \left(1000^x\right)$ h) $\ln \left(\frac{1}{e}\right)$

3. Place each of the following logarithms between two consecutive integers.

   a) $\log_3 60$ b) $\log_2 60$ c) $\log_{10} 2011$

4. Solve each of the following equations for x. Present the exact value of each solution.

   a) $\log_5 x = 3$ b) $\frac{1}{2}\log_2 (3p+1) - 1 = 2$ c) $3\ln(w-4) - 5 = 1$

5. Prove that if $\log_a b \neq 0$, then $\log_a b$ and $\log_b a$ are reciprocals of each other.

6. What is the connection between $\log_a b$ and $\log_a \left(\frac{1}{b}\right)$?

Practice Problems

1. Find the exact value of each of the following logarithmic expressions.

a) $\log_2 32$
b) $\log_3 81$
c) $\log_4 4$
d) $\log_5 1$
e) $\log_6\left(\frac{1}{36}\right)$
f) $\log_{49} 7$
g) $\log_8 4$
h) $\log_{10} 0.000001$
i) $\ln\left(\frac{1}{e^2}\right)$
j) $\log_3 9$
k) $\log_3 3$
l) $\log_3 1$
m) $\log_3\left(\frac{1}{3}\right)$
n) $\log_3 0$
o) $\log_3(-3)$
p) $\log_9 3$
q) $\log_{1/3} 9$
r) $\log_7\left(7^{23}\right)$
s) $\log_8\left(\frac{1}{4}\right)$
t) $\log_2 8$
u) $\log_8 2$
v) $\log_{81} 3$
w) $\log_8 16$
x) $\log_{27}\left(\frac{1}{9}\right)$

2. Simplify each of the following expressions.

a) $\log_3\left(81^A\right)$
b) $\log_{\sqrt{6}}\left(6^{10}\right)$
c) $\ln\left(\sqrt[3]{e}\right)$
d) $\log_2\left(8^x\right)$
e) $\log_8\left(2^x\right)$
f) $\log_{\sqrt{2}}\left(\frac{1}{8}\right)$
g) $\log_9\left(\frac{1}{3^x}\right)$
h) $\log_{1000}\left(10^M\right)$
i) $\log_{1/2}\left(8^x\right)$
j) $\log_{1/2}\left(\sqrt{8}\right)$
k) $\ln\left(\frac{1}{\sqrt[7]{e}}\right)$

3. Place each of the following logarithms between two consecutive integers.

a) $\log_2 10$
b) $\log_5 10$
c) $\log_3 10$
d) $\log_{10} 2011$
e) $\log_{10} 38\,000$
f) $\log_2 100$
g) $\log_3 100$
h) $\log_2\left(\frac{1}{6}\right)$

4. Re-write each of the following logarithmic statements as an exponential statement.

a) $\log_3 x = 10$
b) $\ln A = -2$
c) $\log_5 10 = y$
d) $\log_x y = 4$
e) $\log_A B = C$
f) $\ln(x-2) = 5$

5. Re-write each of the following exponential statements as a logarithmic statement.

a) $A^B = C$ b) $3^x = 8$ c) $2^{t-5} = 11$ d) $e^x = 10$

6. Solve each of the following equations for x. Present the exact value of each solution.

a) $\log_3 x = 2$
b) $\log_2(x-5) = 3$
c) $\ln(x-1) = 2$
d) $\log_7(5x-1) = 2$
e) $\log_3\left(2x^2-5\right) = 3$
f) $\log_{10}(x+2) = -2$
g) $\frac{2}{3}\ln(x-1) - 1 = 5$
h) $\frac{\log_5(x+1) - 1}{3} = -1$

Sample Problems - Answers

1. a) 1 b) 0 c) -2 d) $\frac{1}{2}$ e) $-\frac{4}{3}$ f) 8

2. a) 9 b) $3x$ c) $4a$ d) 18 e) $\frac{x}{2}$ f) $-\frac{1}{2}$ g) $3x$ h) -1

3. a) $3 < \log_3 60 < 4$ b) $5 < \log_2 60 < 6$ c) $3 < \log_{10} 2011 < 4$

4. a) 125 b) 21 c) $e^2 + 4$

5. see solutions

6. see solutions

Practice Problems - Answers

1. a) 5 b) 4 c) 1 d) 0 e) -2 f) $\frac{1}{2}$ g) $\frac{2}{3}$ h) -6 i) -2 j) 2
   k) 1 l) 0 m) -1 n) undefined o) undefined p) $\frac{1}{2}$ q) -2 r) 23
   s) $-\frac{2}{3}$ t) 3 u) $\frac{1}{3}$ v) $\frac{1}{4}$ w) $\frac{4}{3}$ x) $-\frac{2}{3}$

2. a) $4A$ b) 20 c) $\frac{1}{3}$ d) $3x$ e) $\frac{x}{3}$ f) -6 g) $-\frac{x}{2}$ h) $\frac{M}{3}$ i) $-3x$ j) $-\frac{3}{2}$ k) $-\frac{1}{7}$

3. a) $3 < \log_2 10 < 4$ b) $1 < \log_5 10 < 2$ c) $2 < \log_3 10 < 3$ d) $3 < \log 2010 < 4$
   e) $4 < \log_{10} 38\,000 < 5$ f) $6 < \log_2 100 < 7$ g) $4 < \log_3 100 < 5$ h) $-3 < \log_2 \left(\frac{1}{6}\right) < -2$

4. a) $3^{10} = x$ b) $e^{-2} = A$ c) $5^y = 10$ d) $x^4 = y$ e) $A^C = B$ f) $e^5 = x - 2$

5. a) $\log_A C = B$ b) $x = \log_3 8$ c) $t - 5 = \log_2 11$ d) $x = \ln 10$

6. a) 9 b) 13 c) $e^2 + 1$ d) 10 e) $4, -4$ f) -1.99 g) $e^9 + 1$ h) $-\frac{24}{25}$

Sample Problems - Solutions

1. Find the exact value of each of the following logarithmic expressions.

a) $\log_3 3$

Solution: $\log_3 3$ is a number, it is the exponent we need to write over 3 to obtain 3.
In short, $3^? = 3$. The answer is clearly 1, since $3^1 = 3$.

$$\log_3 3 = 1$$

b) $\log_4 1$

Solution: $\log_4 1$ is a number, it is the exponent we need to write over 4 to obtain 1.
In short, $4^? = 1$. The answer is clearly 0, since $4^0 = 1$.

$$\log_4 1 = 0$$

c) $\log_5 \left(\frac{1}{25}\right)$

Solution: $\log_5 \left(\frac{1}{25}\right)$ is a number, it is the exponent we need to write over 5 to obtain $\frac{1}{25}$.
In short, $5^? = \frac{1}{25}$. The answer is clearly -2, since $5^{-2} = \frac{1}{25}$.

$$\log_5 \left(\frac{1}{25}\right) = -2$$

d) $\log_9 3$

Solution: $\log_9 3$ is a number, it is the exponent we need to write over 9 to obtain 3.
In short, $9^? = 3$. We know that 3 is the square root of 9. We re-write this statement as an exponential statement

$$3 = \sqrt{9} = 9^{1/2} \quad \text{and so} \quad \log_9 3 = \frac{1}{2}$$

e) $\log_8 \left(\frac{1}{16}\right)$

Solution: $\log_8 \left(\frac{1}{16}\right)$, it is the exponent we need to write over 8 to obtain $\frac{1}{16}$.
In short, $8^? = \frac{1}{16}$. It looks like we have a problem here, $\frac{1}{16}$ is not a straightforward power of 8. The trick is to realize that they both powers of 2, and express 2 first as a power of 8.

$$8^{1/3} = 2 \quad \text{and we also know that} \quad \frac{1}{16} = 2^{-4}$$

$$\text{our plan:} \qquad 8 \longrightarrow 2 \longrightarrow \frac{1}{16}$$

$$\begin{aligned} 8^{1/3} &= 2 && \text{raise both sides to the power of } -4 \\ \left(8^{1/3}\right)^{-4} &= 2^{-4} && \text{recall that } (a^n)^m = a^{nm} \\ 8^{-4/3} &= \frac{1}{16} && \text{and so} \quad \log_8 \left(\frac{1}{16}\right) = -\frac{4}{3} \end{aligned}$$

f) $\ln e^8$

Solution: ln is short for $\log_e$. So the question is $\log_e \left(e^8\right)$. In order to obtain the product e^8, we will need eigth factors of e. This $\ln e^8 = 8$. (If this problem is confusing because of the unusual number e, try to argue why $\log_3 \left(3^{25}\right)$ must be 25.)

2. Simplify each of the following expressions. Assume that $a > 0$ and $a \neq 1$.

a) $\log_a\left(a^9\right)$

Solution: In order to obtain the product a^9, we will need nine factors of a.

b) $\log_2\left(8^x\right)$

Solution: In order to simplify this expression, we will use the fact that $\log_2\left(2^b\right) = b$. The problem is that the base of the logarithm does not match the base of the exponentiation. We will change the base of the exponentiation by re-writing 8^x as a $2-$power. Clearly, $2^3 = 8$.

$$\log_2\left(8^x\right) = \log_2\left(\left(2^3\right)^x\right) = \log_2\left(2^{3x}\right) = 3x$$

Note that we used the following rule of exponents:

$$\left(a^n\right)^m = a^{nm}$$

when we re-wrote 8^x as 2^{3x}.

c) $\log_2\left(16^a\right)$

Solution: In order to simplify this expression, we will use the fact that $\log_2\left(2^b\right) = b$. The problem is that the base of the logarithm does not match the base of the exponentiation. We will change the base of the exponentiation by re-writing 16^a as a $2-$power. Clearly, $2^4 = 16$.

$$\log_2\left(16^a\right) = \log_2\left(\left(2^4\right)^a\right) = \log_2\left(2^{4a}\right) = 4a$$

Note that we used the rule of exponents $\left(a^n\right)^m = a^{nm}$ when we re-wrote 16^a as 2^{4a}.

d) $\log_{25}\left(5^{36}\right)$

Solution: In order to simplify this expression, we will use the fact that $\log_5\left(5^b\right) = b$. The problem is that the base of the logarithm does not match the base of the exponentiation. We will change the base of the exponentiation by re-writing 5 as a $25-$power. Clearly, $5^2 = 25$, but that is not what we need. We need $25^? = 5$. That missing exponent is $\frac{1}{2}$, in other words, $25^{1/2} = 5$

$$\log_{25}\left(5^{36}\right) = \log_{25}\left(\left((25)^{1/2}\right)^{36}\right) = \log_{25}\left(25^{1/2\cdot 36}\right) = \log_{25}\left(25^{18}\right) = 18$$

Note that we used the rule of exponents $\left(a^n\right)^m = a^{nm}$ when we re-wrote 5^{36} as 25^{18}.

e) $\log_{25}\left(5^x\right)$

Solution: In order to simplify this expression, we will use the fact that $\log_5\left(5^b\right) = b$. The problem is that the base of the logarithm does not match the base of the exponentiation. We will change the base of the exponentiation by re-writing 5 as a $25-$power. Clearly, $5^2 = 25$, but that is not what we need. We need $25^? = 5$. That missing exponent is $\frac{1}{2}$, in other words, $25^{1/2} = 5$

$$\log_{25}\left(5^x\right) = \log_{25}\left(\left((25)^{1/2}\right)^x\right) = \log_{25}\left(25^{1/2\cdot x}\right) = \log_{25}\left(25^{x/2}\right) = \frac{x}{2}$$

Note that we used the rule of exponents $\left(a^n\right)^m = a^{nm}$ when we re-wrote 5^x as $25^{x/2}$.

f) $\ln\left(\frac{1}{\sqrt{e}}\right)$

Solution: recall that $\ln x$ is short for $\log_e x$ (although nobody writes it that way) and so

$$\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-1/2}\right) = -\frac{1}{2}$$

by the same rule, $\log_a\left(a^b\right) = b$.

g) $\log_{10}\left(1000^x\right)$

Solution: $\log_{10}\left(1000^x\right) = \log_{10}\left(\left(10^3\right)^x\right) = \log_{10}\left(10^{3x}\right) = 3x$

h) $\ln\left(\frac{1}{e}\right)$

Solution: $\ln\left(\frac{1}{e}\right) = \ln\left(e^{-1}\right) = -1$

3. Place each of the following logarithms between two consecutive integers.

a) $\log_3 60$

Solution: We need to place 60 between two consecutive three-powers. Those are 27 and 81. Since 60 is between 27 and 81, $\log_3 60$ is between 3 and 4.

$$27 < 60 < 81 \implies 3^3 < 60 < 3^4 \implies 3 < \log_3 60 < 4$$

b) $\log_2 60$

Solution: We need to place 60 between two consecutive two-powers. Those are 32 and 64. Since 60 is between 32 and 64, $\log_2 60$ is between 5 and 6.

$$\begin{aligned} 32 &< 60 < 64 \\ 2^5 &< 60 < 2^6 \\ 5 &< \log_2 60 < 6 \end{aligned}$$

c) $\log_{10} 2011$

Solution: We need to place 2011 between two consecutive ten-powers. Those are 1000 and 10 000. Since 2011 is between 1000 and 10 000, $\log_{10} 2011$ is between 3 and 4.

$$\begin{aligned} 1000 &< 2011 < 10\,000 \\ 10^3 &< 2011 < 10^4 \\ 3 &< \log_{10} 2011 < 4 \end{aligned}$$

4. Solve each of the following equations for x. Present the exact value of each solution.

a) $\log_5 x = 3$

Solution: We re-write the logarithmic statement as an exponential statement.

$$\begin{aligned} \log_5 x &= 3 \\ 5^3 &= x \\ 125 &= x \end{aligned}$$

We check: $\log_5 125 = 3$ is indeed true.

b) $\frac{1}{2}\log_2(3p+1)-1=2$

Solution: We first isolate the logarithmic expression. Then we re-write the logarithmic statement as an exponential statement, and solve for p.

$$\begin{aligned}
\frac{1}{2}\log_2(3p+1)-1 &= 2 && \text{add 1}\\
\frac{1}{2}\log_2(3p+1) &= 3 && \text{multiply by 2}\\
\log_2(3p+1) &= 6 && \text{re-write into an exponential statement}\\
3p+1 &= 2^6 &&\\
3p+1 &= 64 && \text{subtract 1}\\
3p &= 63 && \text{divide by 3}\\
p &= 21 &&
\end{aligned}$$

We check: if $p=21$, then

$$\text{LHS}=\frac{1}{2}\log_2(3\cdot 21+1)-1=\frac{1}{2}\log_2(63+1)-1=\frac{1}{2}(\log_2 64)-1=\frac{1}{2}\cdot 6-1=3-1=2=\text{RHS}$$

c) $3\ln(w-4)-5=1$

Solution: We first isolate the logarithmic expression. Then we re-write the logarithmic statement as an exponential statement, and solve for w.

$$\begin{aligned}
3\ln(w-4)-5 &= 1 && \text{add 5}\\
3\ln(w-4) &= 6 && \text{divide by 3}\\
\ln(w-4) &= 2 && \text{re-write into an exponential statement}\\
w-4 &= e^2 && \text{add 4}\\
w &= e^2+4 &&
\end{aligned}$$

We check: if $w=e^2+4$, then

$$\text{LHS}=3\ln\left(\left(e^2+4\right)-4\right)-5=3\ln\left(e^2\right)-5=3\cdot 2-5=1=\text{RHS}$$

5. Claim: If $\log_a b\neq 0$, then $\log_a b$ and $\log_b a$ are reciprocals of each other.
Proof. Let $x=\log_a b$ Then the statement can be re-written as $a^x=b$.

$$\begin{aligned}
x &= \log_a b &&\\
a^x &= b && \text{raise both sides to the power of } \frac{1}{x}\\
(a^x)^{1/x} &= b^{1/x} && (a^x)^{1/x}=a^{x\cdot\frac{1}{x}}=a^1=a\\
a &= b^{1/x} && \text{same as } \frac{1}{x}=\log_b a\\
\frac{1}{x} &= \log_b a &&
\end{aligned}$$

6. $\log_a b$ and $\log_a \left(\frac{1}{b}\right)$ are opposites of each other.

Proof. Let $x = \log_a b$ Then the statement can be re-written as $a^x = b$.

$$\begin{aligned} x &= \log_a b \\ a^x &= b && \text{raise both sides to the power of } -1 \\ (a^x)^{-1} &= b^{-1} && (a^x)^{-1} = a^{x\cdot(-1)} = a^{-x} \\ a^{-x} &= \frac{1}{b} && \text{same as} \quad -x = \log_a \left(\frac{1}{b}\right) \\ -x &= \log_a \left(\frac{1}{b}\right) \end{aligned}$$