Definition: **The symbol $\log_a b$ represents the number that if we write as an exponent of a, we achieve b.** This expression is only meaningful if both a and b are positive numbers and $a \neq 1$.

Every logarithmic statement can be re-written as an exponential statement.

$$\log_a b = x \text{ is the same as } a^x = b$$

Example: Find the value of $\log_2 8$.
$\log_2 8$ is a number, it is the exponent we need to write over 2 to obtain 8. In short, $2^? = 8$. The answer is clearly 3, since $2^3 = 8$.

$$\log_2 8 = 3$$

Similarly, most exponential statements can be re-written as a logarithmic statement.

$$n^y = M \text{ is the same as } \log_n M = y$$

Some properties of logarithms follow immediately from its definition.

Theorem 1. $\log_a \left(a^k\right) = k$

Proof. To achieve a^k, we need to raise a exactly to the kth power. Or, using exponential notation,

$$\text{if } x = \log_a a^k, \quad \text{then } a^x = a^k \quad \Longrightarrow \quad x = k$$

Theorem 2. $a^{\log_a b} = b$

Proof. Recall the definition of $\log_a b$. This is the number we need to use as an exponent of a to obtain b. In other words,

$$\text{if } x = \log_a b, \quad \text{then } a^x = b \quad \Longrightarrow \quad a^{\log_a b} = b$$

Sample Problems

Simplify each of the following expressions. Assume that $a, X > 0$ and $a \neq 1$.

1. $\log_a \left(a^9\right)$
2. $8^{\log_2 5}$
3. $16^{\log_4 a}$
4. $5^{\log_{25} 3}$
5. $2^{\log_8 a}$
6. $e^{-5 \ln B}$
7. $10^{3 \log_{10} X}$
8. $10^{M \log_{100} 3}$

Practice Problems

Simplify each of the following expressions. Assume that $m > 0$ and $m \neq 1$.

1. $3^{\log_3 2}$
2. $6^{\log_6 m}$
3. $e^{\ln 12}$
4. $8^{\log_2 7}$
5. $0.1^{\log_{10} m}$
6. $25^{\log_5 (2m)}$
7. $3^{\log_9 m}$
8. $1000^{\log_{10} m}$
9. $\left(\frac{1}{2}\right)^{\log_4 m}$
10. $8^{\log_{(1/2)} m}$
11. $\left(\frac{1}{e}\right)^{\ln 9}$
12. $e^{-\ln(3m)}$

Sample Problems - Answers

1. 9 2. 125 3. a^2 4. $\sqrt{3}$ 5. $\sqrt[3]{a}$ 6. $\dfrac{1}{B^5}$ 7. X^3 8. $\left(\sqrt{3}\right)^M = 3^{\frac{M}{2}}$

Practice Problems - Answers

1. 2 2. m 3. 12 4. 343 5. $\dfrac{1}{m}$ 6. $4m^2$ 7. $\sqrt{m}$ 8. m^3 9. $\dfrac{1}{\sqrt{m}}$ 10 $\dfrac{1}{m^3}$ 11. $\dfrac{1}{9}$ 12. $\dfrac{1}{3m}$

Sample Problems - Solutions

Simplify each of the following expressions. Assume that $a > 0$ and $a \neq 1$.

1. $\log_a\left(a^9\right)$

   Solution: In order to obtain the product a^9, we will need nine factors of a.

2. $8^{\log_2 5}$

   Solution: In order to simplify this expression, we will use the fact that $2^{\log_2 5} = 5$. Our expression is not exactly the same as $2^{\log_2 5}$ but it is related to it, since $8 = 2^3$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$8^{\log_2 5} = \left(2^3\right)^{\log_2 5} = 2^{3\log_2 5} = 2^{(\log_2 5)\cdot 3} = \left(2^{\log_2 5}\right)^3 = 5^3 = 125$$

3. $16^{\log_4 a}$

   Solution: In order to simplify this expression, we will use the fact that $4^{\log_4 a} = a$. Our expression is not exactly the same as $4^{\log_4 a}$ but it is related to it, since $16 = 4^2$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$16^{\log_4 a} = \left(4^2\right)^{\log_4 a} = 4^{2\log_4 a} = 4^{(\log_4 a)\cdot 2} = \left(4^{\log_4 a}\right)^2 = a^2$$

4. $5^{\log_{25} 3}$

   Solution: In order to simplify this expression, we will use the fact that $25^{\log_{25} 3} = 3$. Our expression is not exactly the same as $25^{\log_{25} 3}$ but it is related to it, since $5 = 25^{1/2}$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$5^{\log_{25} 3} = \left(25^{1/2}\right)^{\log_{25} 3} = 25^{(1/2)\log_{25} 3} = 25^{(\log_{25} 3)(1/2)} = \left(25^{\log_{25} 3}\right)^{1/2} = 3^{1/2} = \sqrt{3}$$

5. $2^{\log_8 a}$

Solution: In order to simplify this expression, we will use the fact that $8^{\log_8 a} = a$. Our expression is not exactly the same as $8^{\log_8 a}$ but it is related to it, since $2 = 8^{1/3}$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$2^{\log_8 a} = \left(8^{1/3}\right)^{\log_8 a} = 8^{(1/3)\log_8 a} = 8^{(\log_8 a)(1/3)} = \left(8^{\log_8 a}\right)^{1/3} = a^{1/3} = \sqrt[3]{a}$$

6. $e^{-5\ln B}$

Solution: In order to simplify this expression, we will use the fact that $e^{\ln B} = B$. Our expression is not exactly the same as $e^{\ln B}$ but it is related to it. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$e^{-5\ln B} = e^{\ln B\cdot(-5)} = \left(e^{\ln B}\right)^{-5} = B^{-5} = \frac{1}{B^5}$$

7. $10^{3\log_{10} X}$

Solution: In order to simplify this expression, we will use the fact that $10^{\log_{10} X} = X$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$10^{3\log_{10} X} = 10^{\log_{10} X\cdot(3)} = \left(10^{\log_{10} X}\right)^3 = X^3$$

8. $10^{M\log_{100} 3}$

Solution: In order to simplify this expression, we will use the fact that $100^{\log_{100} 3} = 3$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

We also have to re-write 10 as a power of 100. Clearly, $10 = 100^{\frac{1}{2}}$

$$10^{M\log_{100} 3} = \left(100^{1/2}\right)^{M\log_{100} 3} = 100^{\frac{1}{2}M\log_{100} 3} = \left(100^{\log_{100} 3}\right)^{\frac{1}{2}M} = \left(3^{\frac{1}{2}}\right)^M = \left(\sqrt{3}\right)^M = 3^{\frac{M}{2}}$$

