Sample Problems

1. In each of the parametric equations given, find the value of the parameter m so that the equation has exactly one real solution.

   a) $x^2 + mx - 4m = 0$ b) $mx^2 + 2mx + m = x + 2$ c) $2x + \frac{m}{x} = 1$

2. Consider the parametric equation $5m - 3x - 5mx + 6m^2 + x^2 - 4 = 0$.

   a) Find all values of m for which $x = 7$ is a solution of the equation.

   b) Find all values of m for which there are two different real solutions of this equation.

   c) Find all values of the parameter m for which the two solutions of the equation add up to 23.

   d) Find all values of the parameter m for which the product of the two solutions of the equation is 10.

3. Consider the equation $x + mx + 3mx^2 + 1 = 5x^2$.

   a) Find all values of m for which the equation has exactly one real solution.

   b) Find all values of m for which the equation has no real solution.

   c) Find all values of m for which the two real solutions x_1 and x_2 are such that $x_1^2 + x_2^2 = \frac{1}{4}$.

4. Consider the parametric equation $m + 2mx + mx^2 = 4x - 12$

   a) Find all values of m for which $x = -2$ is a solution of the equation.

   b) Find all values of m for which there are two real solutions of the equation.

   c) Find all values of m for which there are two real solutions, x_1 and x_2 of the equation such that $x_1^2 + x_2^2 = 58$.

Sample Problems – Answers

1. a) $0, -16$ b) $0, -\frac{1}{4}$ c) $0, \frac{1}{8}$

2. a) $1, 4$ b) $m \neq -5$ c) 4 d) $-2, \frac{7}{6}$

3. a) $\frac{5}{3}, 3, 7$ b) $(3, 7)$ c) -1

4. a) -20 b) $(-\infty, 0) \cup \left(0, \frac{1}{4}\right)$ c) -1

Sample Problems – Solutions

1. In each of the parametric equations given, find the value of the parameter m so that the equation has exactly one solution.

a) $x^2 + mx - 4m = 0$ $0, -16$

Solution 1: Completing the square.

$$\begin{aligned} x^2 + mx - 4m &= 0 \\ \underbrace{x^2 + mx + \frac{m^2}{4}} - \frac{m^2}{4} - 4m &= 0 \\ \left(x + \frac{m}{2}\right)^2 - \frac{m^2}{4} - 4m &= 0 \end{aligned} \qquad \left(x + \frac{m}{2}\right)^2 = x^2 + mx + \frac{m^2}{4}$$

For exactly one solution, we need that $-\frac{m^2}{4} - 4m = 0$. We solve this equation for m.

$$\begin{aligned} -\frac{m^2}{4} - 4m &= 0 \quad \text{multiply by } -4 \\ m^2 + 16m &= 0 \\ m(m + 16) &= 0 \end{aligned}$$

$$m_1 = 0 \qquad m_2 = -16$$

Solution 2: The Quadratic Formula.
Based on our equation, we have $a = 1$, $b = m$, and $c = -4m$. For exactly one solution, we need the discriminant, $b^2 - 4ac$ to be zero. We solve this equation for m.

$$\begin{aligned} b^2 - 4ac &= 0 \\ m^2 - 4(1)(-4m) &= 0 \\ m^2 + 16m &= 0 \\ m(m + 16) &= 0 \end{aligned}$$

$$m_1 = 0 \qquad m_2 = -16$$

b) $mx^2 + 2mx + m = x + 2 \quad 0, -\frac{1}{4}$

Solution 1: Completing the square.

Before we proceed as usual, we have to consider the case for which the equation is linear, namely, $m = 0$. (Remember, we can NOT treat linear equations as a special case of a quadratic equation.) If $m = 0$, our equation is $0 = x + 2$ which indeed has exactly one solution. Now, if $m \neq 0$, we complete the square

$$\begin{aligned}
mx^2 + x(2m-1) + m - 2 &= 0 \\
m\left(x^2 + \frac{2m-1}{m}x + \frac{m-2}{m}\right) &= 0 \\
m\left(x^2 + \frac{2m-1}{m}x + \frac{(2m-1)^2}{4m^2} - \frac{(2m-1)^2}{4m^2} + \frac{m-2}{m}\right) &= 0 \\
m\left(\left(x + \frac{2m-1}{2m}\right)^2 - \frac{(2m-1)^2}{4m^2} + \frac{m-2}{m}\right) &= 0
\end{aligned}$$

To have exactly one solution, the part after the complete square must be equal to zero.

$$\begin{aligned}
-\frac{(2m-1)^2}{4m^2} + \frac{m-2}{m} &= 0 \qquad \text{multiply by } 4m^2 \\
-(2m-1)^2 + 4m(m-2) &= 0 \\
-4m^2 + 4m - 1 + 4m^2 - 8m &= 0 \\
-4m - 1 &= 0 \\
-1 &= 4m \\
-\frac{1}{4} &= m
\end{aligned}$$

Solution 2: The Quadratic Formula.

Before we do as usual, we have to consider the case for which the equation is linear, namely, $m = 0$. (Remember, we can NOT treat linear equations as a special case of the quadratic...) If $m = 0$, our equation is $0 = x + 2$ which indeed has exactly one solution. Now, if $m \neq 0$, we have $mx^2 + x(2m-1) + m - 2 = 0$ and thus $a = m$, $b = 2m - 1$, and $c = m - 2$. To have exactly one solution, we need the discriminant, $b^2 - 4ac$ to be zero. We solve this equation for m.

$$\begin{aligned}
b^2 - 4ac &= 0 \\
(2m-1)^2 - 4(m)(m-2) &= 0 \\
4m^2 - 4m + 1 - 4m^2 + 8m &= 0 \\
4m + 1 &= 0 \\
4m &= -1 \\
m &= -\frac{1}{4}
\end{aligned}$$

c) $2x + \dfrac{m}{x} = 1$ $\quad 0, \dfrac{1}{8}$

Solution 1: Completing the square.

First we separately consider the case for which the equation is linear, namely, $m = 0$. If $m = 0$, our equation is $2x = 1$ which indeed has exactly one solution. Now, if $m \neq 0$, we multiply both sides by x and complete the square

$$\begin{aligned}
2x + \frac{m}{x} &= 1 && \text{multiply by } x \\
2x^2 + m &= x && \text{subtract } x \\
2x^2 - x + m &= 0 && \text{factor out 2} \\
2\left(x^2 - \frac{1}{2}x + \frac{m}{2}\right) &= 0 && \left(x - \frac{1}{4}\right)^2 = x^2 - \frac{1}{2}x + \frac{1}{16} \\
2\left(\underbrace{x^2 - \frac{1}{2}x + \frac{1}{16}} - \frac{1}{16} + \frac{m}{2}\right) &= 0 \\
2\left(\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + \frac{m}{2}\right) &= 0
\end{aligned}$$

To have exactly one solution, the part after the complete square must be equal to zero.

$$\begin{aligned}
-\frac{1}{16} + \frac{m}{2} &= 0 && \text{multiply by 16} \\
-1 + 8m &= 0 \\
8m &= 1 \\
m &= \frac{1}{8}
\end{aligned}$$

There is one more possibility we need to consider. The original equation is $2x + \dfrac{m}{x} = 1$ and NOT $2x^2 - x + m = 0$. We might get exactly one solution as follows: the discriminant is positive, indicating two solutions, but one of the solutions is 0 which would have to be ruled out as it is not in the domain of the original equation. To check out this case, we take $2x^2 - x + m = 0$ and see what m is so that one of the solutions for x is zero.

$$\begin{aligned}
2x^2 - x + m &= 0 \\
2(0)^2 - 0 + m &= 0 \\
m &= 0
\end{aligned}$$

We have already considered this case, and so our solution, $m_1 = 0$, $m_2 = \dfrac{1}{8}$ is complete.

Solution 2: Quadratic Formula

First we separately consider the case for which the equation is linear, namely, $m = 0$. If $m = 0$, our equation is $2x = 1$ which indeed has exactly one solution. Now, if $m \neq 0$, we multiply both sides by x and use the formula:

$$\begin{aligned}
2x + \frac{m}{x} &= 1 && \text{multiply by } x \\
2x^2 + m &= x && \text{subtract } x \\
2x^2 - x + m &= 0
\end{aligned}$$

Now $a = 2$, $b = -1$, and $c = m$. To have exactly one solution, the discriminant needs to be zero. We solve this equation for m.

$$\begin{aligned}
b^2 - 4ac &= 0 & 1 &= 8m \\
(-1)^2 - 4(2)m &= 0 & \frac{1}{8} &= m \\
1 - 8m &= 0
\end{aligned}$$

There is one more possibility we need to consider. The original equation is $2x+\dfrac{m}{x}=1$ and NOT $2x^2-x+m=0$. We might get exactly one solution as follows: the discriminant is positive, indicating two solutions, but one of the solutions is 0 which would have to be ruled out as it is not in the domain of the original equation. To check out this case, we take $2x^2-x+m=0$ and see what m is so that one of the solutions for x is zero.

$$\begin{aligned} 2x^2-x+m &= 0 \\ 2(0)^2-0+m &= 0 \\ m &= 0 \end{aligned}$$

We have already considered this case, and so our solution, $m_1=0$, $m_2=\dfrac{1}{8}$ is complete.

2. Consider the parametric equation $5m-3x-5mx+6m^2+x^2-4=0$.

a) Find all values of m for which $x=7$ is a solution of the equation. $1,4$

Solution: Just plug in $x=7$ into the equation and see what that gives us for m.

$$\begin{aligned} 5m-3x-5mx+6m^2+x^2-4 &= 0 \\ 5m-3(7)-5m(7)+6m^2+(7)^2-4 &= 0 \\ 5m-21-35m+6m^2+49-4 &= 0 \\ 6m^2-30m+24 &= 0 \qquad \text{factor out 6} \\ 6\left(m^2-5m+4\right) &= 0 \\ 6(m-4)(m-1) &= 0 \end{aligned}$$

$$m_1=4 \qquad m_2=1$$

b) Find all values of m for which there are two different solutions of this equation. $m\neq -5$

Solution: We rearrange the polynomial by degrees of the variable x. $x^2+x(-5m-3)+6m^2+5m-4=0$. Now $a=1$, $b=-5m-3$, and $c=6m^2+5m-4$. For two solutions, we need the discriminant, b^2-4ac to be positive.

$$\begin{aligned} b^2-4ac &> 0 \\ (-5m-3)^2-4(1)\left(6m^2+5m-4\right) &> 0 \\ 25m^2+30m+9-24m^2-20m+16 &> 0 \\ m^2+10m+25 &> 0 \\ (m+5)^2 &> 0 \\ m &\neq -5 \end{aligned}$$

c) Find all values of the parameter m for which the two solutions of the equation add up to 23. 4

Solution: the sum of the two solutions is $-\dfrac{b}{a}$.

$$\begin{aligned} -\frac{b}{a} &= 23 \\ -(-5m-3) &= 23 \\ 5m+3 &= 23 \\ 5m &= 20 \\ m &= 4 \end{aligned}$$

d) Find all values of the parameter m for which the product of the two solutions of the equation is 10. $-2, \frac{7}{6}$

Solution: the product of the two solutions is $\frac{c}{a}$. Thus we have

$$\begin{aligned}
\frac{c}{a} &= 10 \\
\frac{6m^2+5m-4}{1} &= 10 \\
6m^2+5m-4 &= 10 \\
6m^2+5m-14 &= 0
\end{aligned}$$

We solve for m using the quadratic formula:

$$\begin{aligned}
m_{1,2} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-5 \pm \sqrt{5^2-4(6)(-14)}}{2(6)} = \frac{-5 \pm \sqrt{25-4(6)(-14)}}{12} \\
&= \frac{-5 \pm \sqrt{25+336}}{12} = \frac{-5 \pm \sqrt{361}}{12} = \frac{-5 \pm 19}{12}
\end{aligned}$$

$$m_1 = \frac{-5+19}{12} = \frac{14}{12} = \frac{7}{6} \qquad m_2 = \frac{-5-19}{12} = \frac{-24}{12} = -2$$

3. Consider the equation $x + mx + 3mx^2 + 1 = 5x^2$.

a) Find all values of m for which the equation has exactly one real solution. $\frac{5}{3}, 3, 7$

Solution: We rearrange the equation and obtain $(3m-5)x^2 + (m+1)x + 1 = 0$. Thus $a = 3m-5$, $b = m+1$, and $c = 1$.

Case 1. There might be one solution if the equation is linear, i.e. if $a = 0$. This happens when

$$\begin{aligned}
3m-5 &= 0 \\
3m &= 5 \\
m &= \frac{5}{3}
\end{aligned}$$

Then the equation becomes. we write $m = \frac{5}{3}$:

$$\begin{aligned}
\frac{8}{3}x + 1 &= 0 \qquad \text{subtract } 1 \\
\frac{8}{3}x &= -1 \qquad \text{divide by } \frac{8}{3} \\
x &= -\frac{3}{8}
\end{aligned}$$

Thus $m = \frac{5}{3}$ gives us one solution.

Case 2. If $a \neq 0$, (when $m \neq \frac{5}{3}$) then the equation is quadratic. It will have exactly one solution when the discriminant, $b^2 - 4ac$ is zero. This will give us an equation in m.

$$\begin{aligned}
b^2 - 4ac &= (m+1)^2 - 4(1)(3m-5) \\
&= m^2 + 2m + 1 - 12m + 20 \\
&= m^2 - 10m + 21 \\
&= (m-3)(m-7)
\end{aligned}$$

$$m_1 = 3 \qquad m_2 = 7$$

Let us check one of these values. If $m = 3$, then the equation is

$$\begin{aligned} (3(3)-5)x^2 + (3+1)x + 1 &= 0 \\ 4x^2+4x+1 &= 0 \\ (2x+1)^2 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

and so we have exactly one real solution. Thus there is exactly one solution for x if $m = \frac{5}{3}$, 3, or 7.

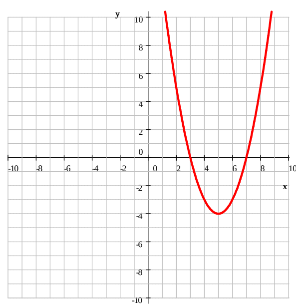
b) Find all values of m for which the equation has no real solution. $(3,7)$

Solution: there is no solution for a quadratic equation when the discriminant is negative. From the previous part, we have that the discriminant, as a function of m is

$$D(m) = (m-3)(m-7)$$

This expression is clearly a quadratic polynomial, with a positive leading coeffiecient. Thus its graph is a regular parabola, which is negative between the two $x-$intercepts. Thus

$$(m-3)(m-7) < 0 \quad \text{if and only if} \quad 3 < m < 7$$



c) Find all values of m for which the two real solutions x_1 and x_2 are such that $x_1^2 + x_2^2 = \frac{1}{4}$. -1

Solution:

$$\begin{aligned} x_1^2 + x_2^2 &= (x_1+x_2)^2 - 2x_1x_2 = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) \\ &= \left(-\frac{m+1}{3m-5}\right)^2 - 2\left(\frac{1}{3m-5}\right) = \frac{(m+1)^2}{(3m-5)^2} - \frac{2}{3m-5} \end{aligned}$$

Thus we need to solve $\frac{1}{4} = \frac{(m+1)^2}{(3m-5)^2} - \frac{2}{3m-5}$ for m Recall that in this case $m \neq \frac{3}{5}$ and so $3m-5 \neq 0$.

$$\begin{aligned} \frac{1}{4} &= \frac{(m+1)^2}{(3m-5)^2} - \frac{2}{3m-5} && \text{mulitply by } 4(3m-5)^2 \\ (3m-5)^2 &= 4(m+1)^2 - 8(3m-5) && \text{subtract 1} \\ 9m^2-30m+25 &= 4\left(m^2+2m+1\right) - 24m + 40 \\ 9m^2-30m+25 &= 4m^2+8m+4-24m+40 \\ 9m^2-30m+25 &= 4m^2-16m+44 \\ 5m^2-14m-19 &= 0 \\ (5m-19)(m+1) &= 0 \end{aligned}$$

$$m_1 = \frac{19}{5} \qquad m_2 = -1$$

Although we obtained two values for m, they will not both work. Since $3 < \frac{19}{5} < 7$, the value $m_1 = \frac{19}{5}$ falls into the interval where there are no real solutions. The only answer is $m = -1$.

4. Consider the parametric equation $m + 2mx + mx^2 = 4x - 12$

a) Find all values of m for which $x = -2$ is a solution of the equation. -20

Solution: We simply substitute $x = -2$ into the equation and solve for m.

$$\begin{aligned} m + 2m(-2) + m(-2)^2 &= 4(-2) - 12 \\ m - 4m + 4m &= -8 - 12 \\ m &= -20 \end{aligned}$$

b) Find all values of m for which there are two real solutions of the equation. $(-\infty, 0) \cup \left(0, \frac{1}{4}\right)$

Solution: We have to be careful with parametric equations if the leading term contains a parameter. We always need to treat the linear case separately. Is there a value of m for which the equation is not quadratic? Clearly, if $m = 0$. We check out this case separately. If $m = 0$, then

$$\begin{aligned} m + 2mx + mx^2 &= 4x - 12 \\ 0 + 2(0)x + (0)x^2 &= 4x - 12 \\ 0 &= 4x - 12 \\ 12 &= 4x \\ 3 &= x \end{aligned}$$

Thus there is exactly one solution for $m = 0$. Now, if $m \neq 0$, the equation is quadratic. Quadratic equations have two different real solutions if the discriminant is positive.

$$\begin{aligned} 0 &= mx^2 + x(2m - 4) + m + 12 \\ D &= b^2 - 4ac = (2m - 4)^2 - 4m(m + 12) = 16 - 64m \end{aligned}$$

$$\begin{aligned} D &> 0 \\ 16 - 64m &> 0 \\ 16 &> 64m \\ \frac{1}{4} &> m \end{aligned}$$

Thus it appears, the solution is all values of m that are less than $\frac{1}{4}$. However, we do need to rule out $m = 0$ since then the equation has only one solution. Thus the answer is $m < \frac{1}{4}$, $m \neq 0$, or, in interval notation: $(-\infty, 0) \cup \left(0, \frac{1}{4}\right)$.

c) Find all values of m for which there are two real solutions, x_1 and x_2 of the equation such that $x_1^2 + x_2^2 = 58$. -1

Solution:

$$\begin{aligned} x_1^2 + x_2^2 &= (x_1 + x_2)^2 - 2x_1x_2 = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a} \\ &= \frac{(2m-4)^2}{m^2} - \frac{2(m+12)}{m} = \frac{4(m-2)^2}{m^2} - \frac{2(m+12)}{m} = \frac{4(m-2)^2}{m^2} - \frac{2m(m+12)}{m^2} \\ &= \frac{4(m-2)^2 - 2m(m+12)}{m^2} = \frac{4m^2 - 16m + 16 - 2m^2 - 24m}{m^2} = \frac{2m^2 - 40m + 16}{m^2} \end{aligned}$$

$$\begin{aligned}
\frac{2m^2-40m+16}{m^2} &= 58 \\
2m^2-40m+16 &= 58m^2 \\
0 &= 56m^2+40m-16 \\
0 &= 8\left(7m^2+5m-2\right) \\
0 &= 8\left(7m-2\right)\left(m+1\right)
\end{aligned}$$

$$m_1=\frac{2}{7} \quad m_2=-1$$

Since $\frac{2}{7}>\frac{1}{4}$, there is no solution for x if $m=\frac{2}{7}$, and thus it is not a solution. On the other hand, -1 falls into the range where the equation has two solutions and so it is correct. We can check: if $m=-1$, then our equation is

$$\begin{aligned}
m+2mx+mx^2 &= 4x-12 \qquad m=-1 \\
-1+2\left(-1\right)x+\left(-1\right)x^2 &= 4x-12 \\
-x^2-2x-1 &= 4x-12 \\
0 &= x^2+6x-11
\end{aligned}$$

Thus $a=1$, $b=6$, and $c=-11$. Since

$$\begin{aligned}
x_1+x_2 &= -\frac{b}{a} \\
x_1x_2 &= \frac{c}{a} \qquad \text{and} \\
x_1^2+x_2^2 &= \left(x_1+x_2\right)^2-2x_1x_2=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right) \\
&= \left(-\frac{6}{1}\right)^2-2\left(\frac{-11}{1}\right)=\left(-6\right)^2-2\left(-11\right)=36+22=58
\end{aligned}$$

