Sample Problems

1.) Find the coordinates of all points where the given parabola and line intersect each other.

a) $\begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$ b) $\begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$ c) $\begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$

2.) Find the coordinates of all points where the given circle and line intersect each other.

a) $\begin{cases} (x-4)^2 + (y+1)^2 = 20 \\ x + 3y = 11 \end{cases}$ c) $\begin{cases} (x-3)^2 + (y-1)^2 = 20 \\ x - y = -5 \end{cases}$

b) $\begin{cases} (x+5)^2 + (y-6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$ d) $\begin{cases} (x+1)^2 + (y-2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$

3.) Find the coordinates of all points where the given circles intersect each other.

$\begin{cases} (x-4)^2 + (y+1)^2 = 20 \\ (x-7)^2 + (y-8)^2 = 50 \end{cases}$

Practice Problems

1.) Find the coordinates of all points where the given parabola and line intersect each other.

a) $\begin{cases} y = x^2 + 4x - 18 \\ y + 3 = 2x \end{cases}$ c) $\begin{cases} y = -x^2 + 3x + 1 \\ y = 7x + 9 \end{cases}$ e) $\begin{cases} y = -\frac{1}{2}x^2 + 3x + 4 \\ y = 5x + 6 \end{cases}$

b) $\begin{cases} y = x^2 + 4x - 12 \\ y = -2x - 21 \end{cases}$ d) $\begin{cases} y = x^2 - 5x - 4 \\ y = x - 12 \end{cases}$ f) $\begin{cases} y = -x^2 - 6x + 1 \\ y = -2x + 9 \end{cases}$

2.) Find the coordinates of all points where the given circle and line intersect each other.

a) $\begin{cases} x^2 + (y+4)^2 = 20 \\ x - 3y = 2 \end{cases}$ d) $\begin{cases} (x-4)^2 + (y-3)^2 = 25 \\ 3x + 4y = 24 \end{cases}$ g) $\begin{cases} (x-1)^2 + (y+7)^2 = 25 \\ y = -\frac{1}{2}x - \frac{3}{2} \end{cases}$

b) $\begin{cases} (x-2)^2 + (y+5)^2 = 10 \\ 3y - x = -7 \end{cases}$ e) $\begin{cases} (x-2)^2 + (y+1)^2 = 20 \\ x - 2y = 19 \end{cases}$ h) $\begin{cases} (x-2)^2 + (y-1)^2 = 25 \\ 3y = 4x - 30 \end{cases}$

c) $\begin{cases} (x-3)^2 + (y+1)^2 = 16 \\ y + x = 12 \end{cases}$ f) $\begin{cases} (x+1)^2 + (y-3)^2 = 50 \\ x - 7y + 72 = 0 \end{cases}$ i) $\begin{cases} (x+1)^2 + (y-4)^2 = 17 \\ x + 4y = 15 \end{cases}$

3.) Find the coordinates of all points where the given circles intersect each other.

a) $\begin{cases} x^2+(y+1)^2=25 \\ (x+5)^2+(y-4)^2=65 \end{cases}$

b) $\begin{cases} (x+1)^2+(y-2)^2=20 \\ (x+2)^2+(y-1)^2=34 \end{cases}$

c) $\begin{cases} (x-3)^2+y^2=50 \\ (x-2)^2+(y+2)^2=45 \end{cases}$

d) $\begin{cases} (x+2)^2+(y-5)^2=10 \\ (x+5)^2+(y-6)^2=40 \end{cases}$

e) $\begin{cases} (x+1)^2+y^2=4 \\ (x+4)^2+(y+3)^2=40 \end{cases}$

Sample Problems - Answers

1.) a) $(-5,-15)$ and $(1,-3)$ b) $(1,-5)$ c) no intersection point

2.) a) $(8,1)$ and $(2,3)$ b) $(-6,3)$ c) no intersection point d) $(3,5)$ and $(-5,-1)$

3.) $(2,3)$ and $(8,1)$

Practice Problems - Answers

1.) a) $(3,3)$ and $(-5,-13)$ b) $(-3,-15)$ c) no intersection point d) $(2,-10)$ and $(4,-8)$

e) $(-2,-4)$ f) no intersection point

2.) a) $(4,-2)$ and $(2,0)$ b) $(1,-2)$ c) no intersection point d) $(0,6)$ and $(8,0)$

e) no intersection point f) $(-2,10)$ g) $(1,-2)$ and $(5,-4)$ h) $(6,-2)$ i) $(3,3)$ and $(-5,5)$

3.) a) $(3,3)$ and $(-4,-4)$ b) $(1,6)$ $(3,4)$ c) $(-4,1)$ $(8,-5)$ d) $(1,4)$

e) no intersection point

Sample Problems - Solutions

1.) Find the coordinates of all points where the given parabola and line intersect each other.

a) $\begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$

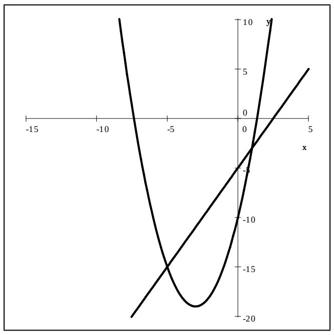
Solution: We will use substitution. We solve for y in the second equation: $y = 2x - 5$. We substitute this into the first equation and solve for x.

$$\begin{aligned} 2x - 5 &= x^2 + 6x - 10 \\ 0 &= x^2 + 4x - 5 \\ 0 &= (x+5)(x-1) \qquad \Longrightarrow \qquad x_1 = -5 \text{ and } x_2 = 1 \end{aligned}$$

The graphs intersect each other in two points. We can find the $y-$coordinates by using either of the equations, so let us use the easier one, that of the line, $y = 2x - 5$.

$$y_1 = 2x_1 - 5 = 2(-5) - 5 = -15 \quad \text{and} \quad y_2 = 2x_2 - 5 = 2(1) - 5 = -3$$

Thus the intersection points are $(-5, -15)$ and $(1, -3)$.



We check: the coordinates of an intersection point must be the solution of both equations. For the point $(-5, -15)$, we check

$$\begin{aligned} y &= x^2 + 6x - 10 \\ \text{RHS} &= x^2 + 6x - 10 = (-5)^2 + 6(-5) - 10 = 25 - 30 - 10 = -15 = \text{LHS} \\ y &= 2x - 5 \\ \text{RHS} &= 2(-5) - 5 = -15 = \text{LHS} \end{aligned}$$

Thus $(-5, -15)$ is indeed an intersection point. We also check the point $(1, -3)$

$$\begin{aligned} y &= x^2 + 6x - 10 \\ \text{RHS} &= x^2 + 6x - 10 = (1)^2 + 6 \cdot 1 - 10 = 1 + 6 - 10 = -3 = \text{LHS} \\ y &= 2x - 5 \\ \text{RHS} &= 2(1) - 5 = -3 = \text{LHS} \end{aligned}$$

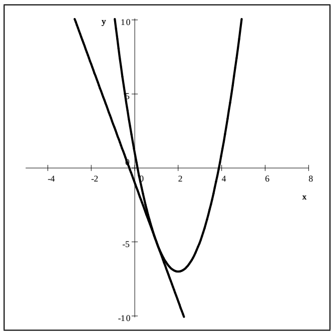
Thus $(1, -3)$ is also an intersection point, and so our solution is correct.

b) $\begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$

Solution: We will use substitution. We substitute $y = -4x - 1$ into the first equation and solve for x.

$$\begin{aligned} -4x - 1 &= 2x^2 - 8x + 1 \\ 0 &= 2x^2 - 4x + 2 \\ 0 &= 2\left(x^2 - 2x + 1\right) \\ 0 &= 2(x-1)^2 \qquad \Longrightarrow \qquad x = 1 \end{aligned}$$

The graphs intersect each other in one point. When that happens, the line is called the **tangent line** drawn to the parabola. We can find the $y-$coordinate by using either of the equations, so let us use the easier one, that of the line, $y = -4x - 1 = -4(1) - 1 = -5$. Thus the intersection point (also called the point of tangency) is $(1, -5)$.



We check: the coordinates of an intersection point must be the solution of both equations.

$$\begin{aligned} y &= 2x^2 - 8x + 1 \\ \text{RHS} &= 2(1)^2 - 8 \cdot 1 + 1 = 2 - 8 + 1 = -5 = \text{LHS} \\ y &= -4x - 1 \\ \text{RHS} &= -4 \cdot 1 - 1 = -5 = \text{LHS} \end{aligned}$$

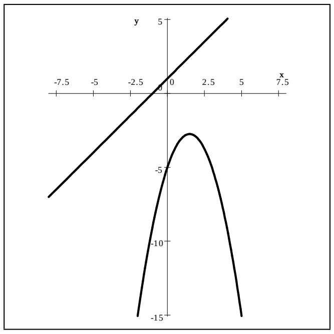
Thus $(1, -5)$ is indeed an intersection point.

c) $\begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$

Solution: We will use substitution. We substitute $y = x + 1$ into the first equation and solve for x.

$$\begin{aligned} x + 1 &= -x^2 + 3x - 5 \\ 0 &= -x^2 + 2x - 6 \\ 0 &= -\left(x^2 - 2x + 6\right) \\ 0 &= -\left[(x-1)^2 + 5\right] \qquad \Longrightarrow \qquad \text{no real solution for } x \end{aligned}$$

The fact that the equation has no solution for x indicates that the graphs do not intersect each other.



2.) Find the coordinates of all points where the given circle and line intersect each other.

a) $\begin{cases} (x-4)^2 + (y+1)^2 = 20 \\ x + 3y = 11 \end{cases}$

Solution: We will use substitution. We solve for x in the second equation: $x = -3y + 11$. We substitute this into the first equation and solve for y.

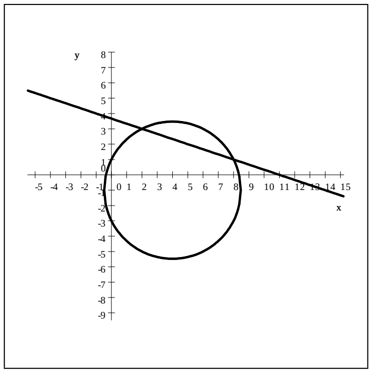
$$\begin{aligned} (x-4)^2 + (y+1)^2 &= 20 \\ (-3y+11-4)^2 + (y+1)^2 &= 20 \\ (-3y+7)^2 + (y+1)^2 &= 20 \\ 9y^2 - 42y + 49 + y^2 + 2y + 1 &= 20 \\ 10y^2 - 40y + 50 &= 20 \end{aligned}$$

$$\begin{aligned} &10y^2 - 40y + 30 = 0 \\ &10\left(y^2 - 4y + 3\right) = 0 \\ &10\,(y-1)\,(y-3) = 0 \\ &y_1 = 1 \text{ and } y_2 = 3 \end{aligned}$$

The graphs intersect each other in two points. We can find the $x-$coordinates by using the equation $x = -3y + 11$.

$$x_1 = -3y_1 + 11 = -3\,(1) + 11 = 8 \quad \text{and} \quad x_2 = -3y_2 + 11 = -3\,(3) + 11 = 2$$

Thus the intersection points are $(8, 1)$ and $(2, 3)$.



We check: the coordinates of an intersection point must be the solution of both equations. For the point $(8, 1)$, we check

$$\begin{aligned} (x-4)^2 + (y+1)^2 &= 20 \\ \text{LHS} &= (8-4)^2 + (1+1)^2 = 4^2 + 2^2 = 16 + 4 = 20 = \text{RHS} \\ x + 3y &= 11 \\ \text{LHS} &= 8 + 3 \cdot 1 = 11 = \text{RHS} \end{aligned}$$

Thus $(8, 1)$ is indeed an intersection point. We also check the point $(2, 3)$

$$\begin{aligned} (x-4)^2 + (y+1)^2 &= 20 \\ \text{LHS} &= (2-4)^2 + (3+1)^2 = (-2)^2 + 4^2 = 4 + 16 = 20 = \text{RHS} \\ x + 3y &= 11 \\ \text{LHS} &= 2 + 3 \cdot 3 = 11 = \text{RHS} \end{aligned}$$

Thus $(2, 3)$ is also an intersection point, and so our solution is correct.

Please note that our choice to solve for x and substitute that was wise. If instead we solved for y, we would have to use the substitution $y = \dfrac{-x+11}{3}$ and thus solve the quadratic equation $(x-4)^2 + \left(\dfrac{-x+11}{3} + 1\right)^2 = 20$ which is much more laborous than what we did.

b) $\begin{cases} (x+5)^2 + (y-6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$

Solution: We will use substitution. We solve for x in the second equation:

$$\begin{aligned} y &= -\frac{1}{3}x + 1 \\ 3y &= -x + 3 \end{aligned} \qquad \begin{aligned} 3y + x &= 3 \\ x &= -3y + 3 \end{aligned}$$

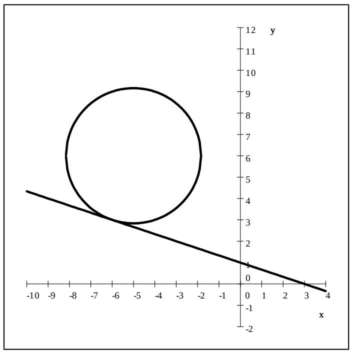
We substitute $x = -3y + 3$ into the first equation and solve for y.

$$\begin{aligned} (x+5)^2 + (y-6)^2 &= 10 \\ (-3y+3+5)^2 + (y-6)^2 &= 10 \\ (-3y+8)^2 + (y-6)^2 &= 10 \\ 9y^2 - 48y + 64 + y^2 - 12y + 36 &= 10 \\ 10y^2 - 60y + 100 &= 10 \end{aligned} \qquad \begin{aligned} & 10y^2 - 60y + 90 = 0 \\ & 10\left(y^2 - 6y + 9\right) = 0 \\ & 10(y-3)^2 = 0 \\ & y = 3 \end{aligned}$$

The graphs intersect each other in one point. When that happens, the line is called the **tangent line** drawn to the circle. We can find the $x-$coordinate by using the equation $x = -3y + 3$.

$$x = -3y + 3 = -3(3) + 3 = -9 + 3 = -6$$

Thus the intersection point is $(-6, 3)$.



We check: the coordinates of an intersection point must be the solution of both equations. For the point $(-6, 3)$, we check

$$\begin{aligned} (x+5)^2 + (y-6)^2 &= 10 \\ \text{LHS} &= (-6+5)^2 + (3-6)^2 = (-1)^2 + (-3)^2 = 1 + 9 = 10 = \text{RHS} \\ y &= -\frac{1}{3}x + 1 \\ \text{RHS} &= -\frac{1}{3}(-6) + 1 = 2 + 1 = 3 = \text{LHS} \end{aligned}$$

Thus $(-6, 3)$ is indeed an intersection point, and so our solution is correct.

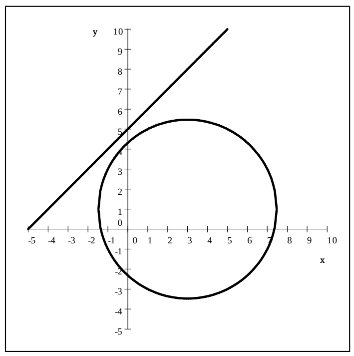
c) $\begin{cases} (x-3)^2 + (y-1)^2 = 20 \\ x - y = -5 \end{cases}$

Solution: We will use substitution. We solve for y in the second equation: $y = x + 5$. We substitute this into the first equation and solve for y.

$$\begin{aligned} (x-3)^2 + (y-1)^2 &= 20 \\ (x-3)^2 + (x+5-1)^2 &= 20 \\ (x-3)^2 + (x+4)^2 &= 20 \\ x^2 - 6x + 9 + x^2 + 8x + 16 &= 20 \end{aligned}$$

$$\begin{aligned} 2x^2 + 2x + 25 &= 20 \\ 2x^2 + 2x + 5 &= 0 \\ 2\left(x^2 + x + \frac{5}{2}\right) &= 0 \\ 2\left(\left(x + \frac{1}{2}\right)^2 + \frac{9}{4}\right) &= 0 \end{aligned}$$

There is no real solution of this equation. This indicates that the graphs do not intersect each other.



d) $\begin{cases} (x+1)^2 + (y-2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$

Solution: We will use substitution. This problem is different from the previous problems because we can not solve for x or y without division, and so the algebra will be slightly more laborous. There are a few tricks to simplify computation in such a situation. We solve for y in the second equation: $y = \dfrac{3x+11}{4}$. We substitute this into the first equation and solve for y.

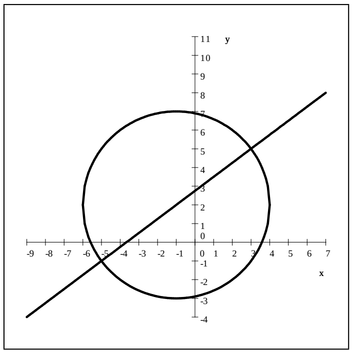
$$\begin{aligned} (x+1)^2 + (y-2)^2 &= 25 \\ (x+1)^2 + \left(\frac{3x+11}{4} - 2\right)^2 &= 25 \\ (x+1)^2 + \left(\frac{3x+11}{4} - \frac{8}{4}\right)^2 &= 25 \\ (x+1)^2 + \left(\frac{3x+3}{4}\right)^2 &= 25 \\ (x+1)^2 + \frac{(3x+3)^2}{4^2} &= 25 \\ (x+1)^2 + \frac{[3(x+1)]^2}{16} &= 25 \\ (x+1)^2 + \frac{9(x+1)^2}{16} &= 25 \end{aligned}$$

$$\begin{aligned} (x+1)^2 + \frac{9}{16}(x+1)^2 &= 25 \\ \frac{25}{16}(x+1)^2 &= 25 \\ (x+1)^2 &= 16 \\ (x+1)^2 - 16 &= 0 \\ (x+1+4)(x+1-4) &= 0 \\ (x+5)(x-3) &= 0 \\ x_1 = -5 \quad \text{and} \quad x_2 &= 3 \end{aligned}$$

The graphs intersect each other in two points. We can find the $x-$coordinates using the equation $y=\dfrac{3x+11}{4}$.

$$y_1=\frac{3x_1+11}{4}=\frac{3(-5)+11}{4}=-1 \quad \text{and} \quad y_2=\frac{3x_2+11}{4}=\frac{3(3)+11}{4}=5$$

Thus the intersection points are $(-5,-1)$ and $(3,5)$.



3.) Find the coordinates of all points where the given circles intersect each other.

$$\begin{cases}(x-4)^2+(y+1)^2=20\\(x-7)^2+(y-8)^2=50\end{cases}$$

Solution: We will first expand the four complete squares and combine like terms and then subtract the two equations. This will achieve a complete cancellation of the quadratic terms, x^2 and y^2. The first equation becomes:

$$\begin{aligned}(x-4)^2+(y+1)^2&=20\\x^2-8x+16+y^2+2y+1&=20 &&\text{subtract } (16+1)=17\\x^2-8x+y^2+2y&=3\end{aligned}$$

The second equation:

$$\begin{aligned}(x-7)^2+(y-8)^2&=50\\x^2-14x+49+y^2-16y+64&=50 &&\text{subtract } (64+49)=113\\x^2-14x+y^2-16y&=-63\end{aligned}$$

So our system now looks like this:

$$\begin{cases}x^2-8x+y^2+2y=3\\x^2-14x+y^2-16y=-63\end{cases}$$

Next, we will subtract the second equation from the first one by adding the opposite. So we multiply the second equation by -1. Then add.

$$\begin{cases}x^2-8x+y^2+2y=3\\-x^2+14x-y^2+16y=63\end{cases}$$

$$\begin{aligned}6x+18y&=66 &&\text{divide by 6}\\x+3y&=11\end{aligned}$$

We can now use this nice linear connection between x and y to solve for one in terms of the other. If we wanted to avoid having to deal with fractions, then solving for x is a much better option than solving for y.

$$x=-3y+11$$

We substitute this information into one of the original equations. We select the first one because it involves smaller numbers.

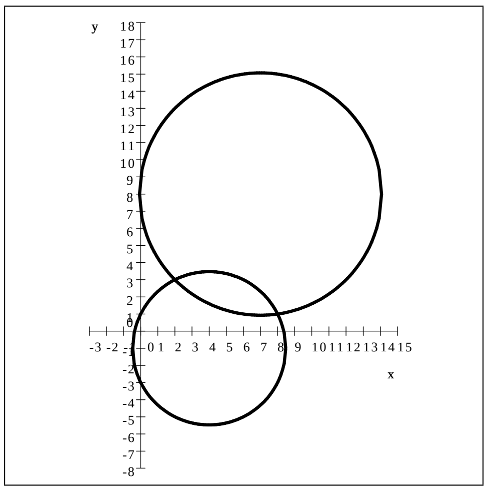
$$\begin{aligned} (x-4)^2+(y+1)^2 &= 20 \\ (-3y+11-4)^2+(y+1)^2 &= 20 \\ (-3y+7)^2+(y+1)^2 &= 20 \\ 9y^2-42y+49+y^2+2y+1 &= 20 \\ 10y^2-40y+50 &= 20 \end{aligned}$$

$$10y^2-40y+30=0$$
$$10\left(y^2-4y+3\right)=0$$
$$10(y-1)(y-3)=0$$
$$y_1=1 \text{ and } y_2=3$$

The graphs intersect each other in two points. We can find the x−coordinates by using the equation $x=-3y+11$.

$$x_1=-3y_1+11=-3(1)+11=8 \quad \text{and} \quad x_2=-3y_2+11=-3(3)+11=2$$

Thus the intersection points are $(8,1)$ and $(2,3)$.



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