The Law of Sines is used to compute angles and sides in triangles that may not have a right angle.

Theorem: (The Law of Sines) If a, b, and c are the three sides of a triangle with corresponding angles α, β, and γ, then

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof: Suppose that a, b, and c are the three sides of a triangle with corresponding angles α, β, and γ. Recall the theorem that the area of the triangle can be computed as $A = \frac{1}{2}ab\sin\gamma$.

We can apply this theorem to the same triangle using different pairs of sides and the angle between them.

$$A = \frac{1}{2}ab\sin\gamma \qquad \text{and} \qquad A = \frac{1}{2}bc\sin\alpha \qquad \text{and} \qquad A = \frac{1}{2}ac\sin\beta$$

We computed the area of the same triangle, and so the area must be the same in each of the cases. Looking at the first and second equation, we get that

$$\begin{aligned} \frac{1}{2}ab\sin\gamma &= \frac{1}{2}bc\sin\alpha && \text{multiply both sides by } 2 \\ ab\sin\gamma &= bc\sin\alpha && \text{divide both sides by } b \\ a\sin\gamma &= c\sin\alpha && \text{divide both sides by } \sin\gamma\sin\alpha \\ \frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma} \end{aligned}$$

Looking at the first and third equation, we get that

$$\begin{aligned} \frac{1}{2}ab\sin\gamma &= \frac{1}{2}ac\sin\beta && \text{multiply both sides by } 2 \\ ab\sin\gamma &= ac\sin\beta && \text{divide both sides by } a \\ b\sin\gamma &= c\sin\beta && \text{divide both sides by } \sin\beta\sin\alpha \\ \frac{b}{\sin\beta} &= \frac{c}{\sin\gamma} \end{aligned}$$

Thus we have proved that all three of $\frac{a}{\sin\alpha}$ and $\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}$ are equal.

The Law of Sines can be used to solve for sides or angles in triangles that may not be right triangles.

Definition: Given some sides and/or angles in a triangle, to **solve a triangle** means finding all missing sides and angles.

Example: Solve the triangle $c = 3\,\mathrm{m}$, $\gamma = 43°$, and $\alpha = 81°$. Present approximate values of all answer, accurate up to three or four decimals.

Solution: We can easily find β since the other two angles were given.

$$\beta = 180° - (\alpha + \gamma) = 180° - (43° + 81°) = 180° - (43° + 81°) = 56°$$

We now know all three angles, which means that we know the triangle up to similarity. Since one side, namely c was also given, this uniquely deterines the triangle. We will use the law of sines to find the missing sides, a and b.

$$\begin{aligned} \frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma} && \text{solve for } c && a = \frac{c\sin\alpha}{\sin\gamma} = \frac{(3\,\mathrm{m})\sin 81°}{\sin 43°} \approx 4.34468\,\mathrm{m} && \text{and similarly,} \\ \frac{b}{\sin\beta} &= \frac{c}{\sin\gamma} && \text{solve for } b && b = \frac{c\sin\beta}{\sin\gamma} = \frac{(3\,\mathrm{m})\sin 56°}{\sin 43°} \approx 3.646\,8016\,\mathrm{m} \end{aligned}$$

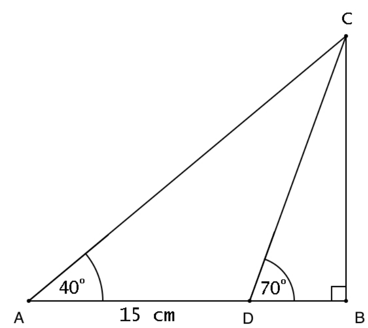
Thus the solution is:

$$\begin{aligned} a &\approx 4.34468\,\mathrm{m}, \quad b \approx 3.646\,8016\,\mathrm{m}, \quad c = 3\,\mathrm{m} \\ \alpha &= 81^\circ, \qquad \beta = 56^\circ, \qquad \text{and} \quad \gamma = 37^\circ \end{aligned}$$

It is good practice to check whether the order between the sides is the same as the order between corresponding angles. Indeed, the shortest side is c and the smallest angle is γ, and the longest side is a and the greatest angle is α.

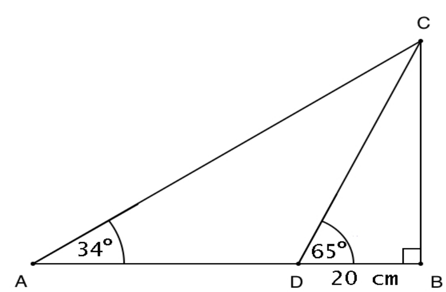
Sample Problems

1. Prove that the area of a triangle can be computed as $A = \dfrac{1}{2} ab \sin \gamma$.
2. Prove the Law of Sines using results from problem 1.
3. Solve the triangle given that $a = 3$, $\alpha = 42^\circ$, and $\beta = 100^\circ$. Present a decimal approximation of all answers, accurate up to four or more decimal places.
4. Compute the length of line segment BD based on the picture given.



Practice Problems

1. Solve each of the following triangles.
   a) $\alpha = 106^\circ$, $\beta = 21^\circ$, $b = 2.4\,\mathrm{ft}$
   b) $\alpha = 62^\circ$, $\gamma = 41^\circ$, $a = 15\,\mathrm{cm}$
2. Compute the length of line segment AC based on the picture given.



Answers

Sample Problems

1. see solutions 2. see solutions 3. $\gamma = 38°$, $b \approx 4.41532$ unit, $c \approx 2.760275$ unit 4. $6.59539\,\mathrm{cm}$

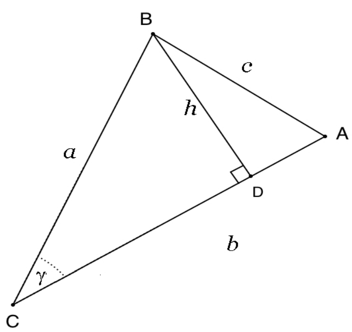
Practice Problems

1. a) $\gamma = 53°$, $a \approx 6.4376\,\mathrm{ft}$, $c \approx 5.348\,484\,\mathrm{ft}$ b) $\beta = 77°$, $b \approx 16.55314\,\mathrm{cm}$, $c \approx 11.1455\,\mathrm{cm}$ 2. $76.7\,\mathrm{cm}$

Solutions – Sample Problems

1. Prove that the area of a triangle can be computed as $A = \dfrac{1}{2}ab\sin\gamma$.

Proof: Consider the picture shown.



Let h denote the height belonging to side b. The area of the triangle can be computed as $\dfrac{1}{2}bh$. Consider now the right triangle BCD. In this triangle, $\sin\gamma = \dfrac{h}{a}$. We solve for h: $h = a\sin\gamma$.

$$A = \frac{1}{2}bh = \frac{1}{2}b\,(a\sin\gamma) = \frac{1}{2}ab\sin\gamma$$

2. Prove the Law of Sines using results from problem 1.

Proof: Consider triangle ABC. We compute the area of the triangle using the same formula but applying it to different sides.

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

We now state just one such equality and cancel out a few things.

$$\begin{aligned} \frac{1}{2}ab\sin\gamma &= \frac{1}{2}ac\sin\beta && \text{divide by } \frac{1}{2}a \\ b\sin\gamma &= c\sin\beta && \text{divide by } bc \\ \frac{\sin\gamma}{c} &= \frac{\sin\beta}{b} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}ac\sin\beta &= \frac{1}{2}bc\sin\alpha && \text{divide by } \frac{1}{2}c \\ a\sin\beta &= b\sin\alpha && \text{divide by } ab \\ \frac{\sin\beta}{b} &= \frac{\sin\alpha}{a} \end{aligned}$$

Thus we have proved that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

3. Solve the triangle given that $a = 3$, $\alpha = 42°$, and $\beta = 100°$. Present a decimal approximation of all answers, accurate up to four or more decimal places.

Solution: Since two angles were given, we can easily compute the third one.

$$\alpha + \beta + \gamma = 180° \quad \Longrightarrow \quad \gamma = 180° - (\alpha + \beta) = 180° - (42° + 100°) = 38°$$

Now that we know all angles and one side, the Law of Sines will enable us to find the length of all sides. To compute b, we state the Law of Sines for a, b, and α and β and solve for b. To make the algebra easier, we will chose a form of the theorem where the unknown, b is in the numerator.

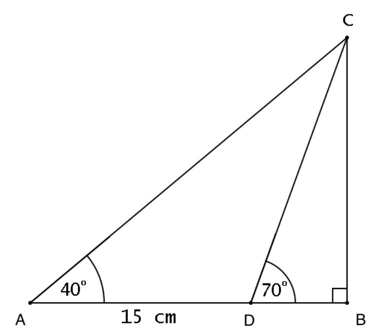
$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \quad \Longrightarrow \quad b = \frac{a\sin\beta}{\sin\alpha} = \frac{3\sin 100°}{\sin 42°} \approx 4.415\,316\,28$$

We find c similarly: we state the Law of Sines for a, c, and α and γ and solve for c.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \quad \Longrightarrow \quad c = \frac{a\sin\gamma}{\sin\alpha} = \frac{3\sin 38°}{\sin 42°} \approx 2.760\,275$$

To solve a triangle means finding all sides and angles that were not given. Thus the answer is: $\gamma = 38°$, $b \approx 4.415\,316\,28$ unit, and $c \approx 2.760\,275$ unit.

4. Compute the length of line segment BD based on the picture below.



Solution: Angle $ADC = 110°$. Now we can compute every side and angle in triangle ADC using the Law of Sines.

$$\measuredangle ACD = 180° - (40° + 110°) = 30°$$

We will now compute the length of line segment CD.

$$\frac{CD}{\sin 40°} = \frac{15\,\text{cm}}{\sin 30°} \quad \Longrightarrow \quad CD = \frac{15\,\text{cm}\sin 40°}{\sin 30°} \approx 19.2836283\,\text{cm}$$

We will now use right triangle trigonometry to compute the length of line segment BD.

$$\cos 70° = \frac{BD}{CD} \quad \Longrightarrow \quad BD = CD\cos 70° \approx 19.2836283\,\text{cm}\cos 70° \approx 6.59539\,\text{cm}$$

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