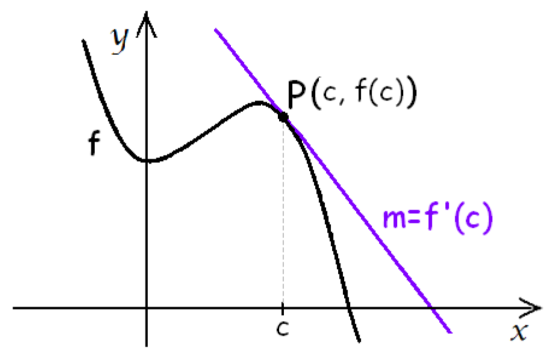
Recall the point-slope equation of a line. Suppose a straight line has slope m and passes through the point $P(a,b)$. The **point-slope equation** of the line is $m(x-a) = y-b$.

Recall that the derivative measures the slope of the tangent line. If f is a function, differentiable at c, then the tangent line drawn to the graph of f at $x=c$ has slope $f'(c)$. And since the tangent line will obviously contain the point of tangency, it is also a line that passes through the point $(c, f(c))$.

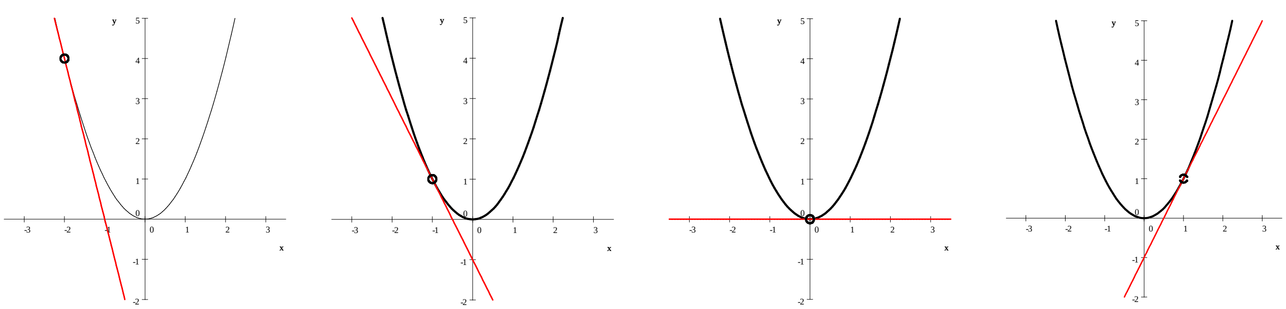


Example 1. Let $f(x) = x^2$. Sketch the tangent line drawn to the graph of f at

a) $x=-2$ b) $x=-1$ c) $x=0$ d) $x=1$

Solution: The derivative measures the slope of the tangent line. The derivative of $f(x)=x^2$ is $f'(x)=2x$, and this is the slope of the tangent line drawn to the graph of f at x.

a) $x=-2$ b) $x=-1$ c) $x=0$ d) $x=1$

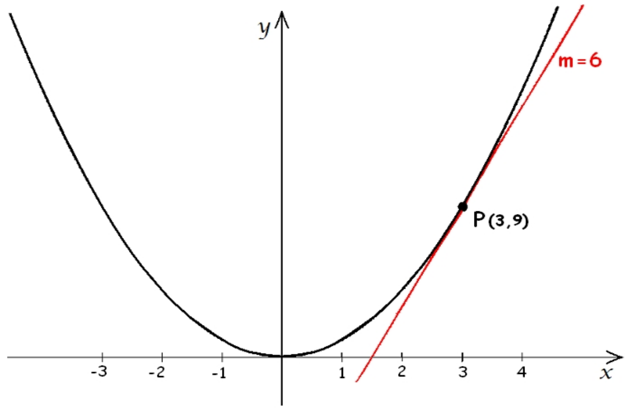


$f'(x)=2x$	$f'(x)=2x$	$f'(x)=2x$	$f'(x)=2x$
$f'(-2)=-4$	$f'(-1)=-2$	$f'(0)=0$	$f'(1)=2$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
$m=-4$	$m=-2$	$m=0$	$m=2$

Example 2. Let $f(x)=x^2$. Find an equation for the tangent line drawn to the graph of f at $x=3$.

Solution: We are asked to find the equation of a line. In this case, we need a point on the line and the slope of the line. For the point, we will use the point of tangency. Clearly, the point of tangency must be on the tangent line. This point is $(3, f(3))$. We evaluate f at $x=3$ and obtain $f(3)=9$. Thus, the point $(3,9)$ is on the tangent line. As for the slope of the tangent line, that is exactly what the derivative measures. At $x=3$, the slope of the tangent line is $m=f'(3)$. Since $f'(x)=2x$, the tangent line will have slope $m=f'(3)=2\cdot 3=6$.

We now have all what we need: the tangent line has slope 6 and passes through the point $(3,9)$. Thus the tangent line's equation is $6(x-3)=y-9$. We can simplify this to obtain the intercept-slope form, $y=6x-9$.



Example 3. Find an equation for the tangent line drawn to graph of $f(x) = \sqrt{x}$ at $x = 9$.

Solution: For the equation of a line, we need a point and the line's slope. The point of tangency, $(9, f(9))$ is obviously on the tangent line. Since $f(9) = \sqrt{9} = 3$, this point is $(9, 3)$. For the slope of the tangent line, we obtain the derivative. The derivative at x is the slope of the tangent line. In short, $m = f'(9)$.

We differentiate $f(x) = \sqrt{x} = x^{1/2}$. By the power rule,

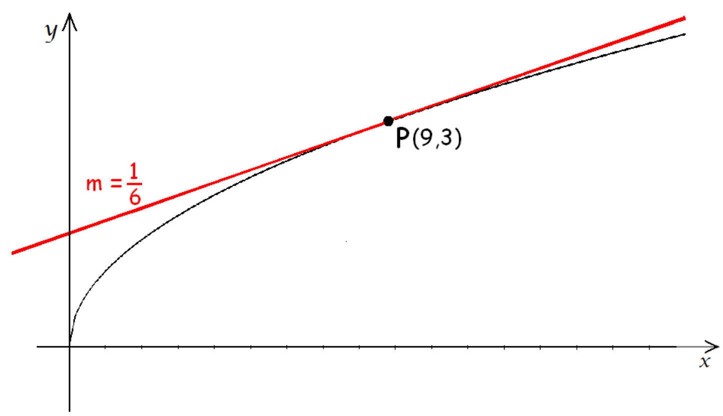
$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

We evaluate this at $x = 3$, and obtain

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

So the tangent line has slope $\frac{1}{6}$ and passes through the point $(9, 3)$. We can write the point-slope form of the equation and transform it to the slope-intercept form.

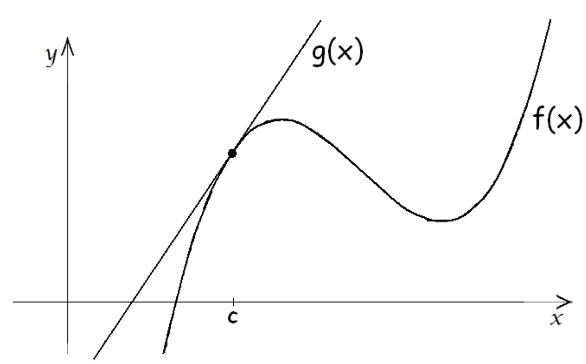
$$y - 3 = \frac{1}{6}(x - 9) \quad \text{or} \quad y = \frac{1}{6}x + \frac{3}{2}$$



Another way of looking at the tangent line drawn to the graph of f as another function g that locally agrees with f up to two levels. If f is a function and g is the tangent line drawn to f at $x = c$, then

$$f(c) = g(c) \quad \text{and} \quad f'(c) = g'(c)$$

That is to say that if we think in terms of location functions, then f and g are location functions of two objects that at c have the same location and the same velocity.



Example 4. Find all values of a and b if we know that $y = -5x + 20$ is a tangent line drawn to the graph of $f(x) = ax^3 + bx^2 - x + 8$ at $x = 2$.

Solution: Let us denote the tangent line by $g(x)$. As stated before, we can think of the tangent line as a second function, that, at $x = 2$, agree with f up to two levels.

$$f(2) = g(2) \quad \text{and } f'(2) = g'(2)$$

We hope that these two statements give us two equations in a and b so we can solve for those.

$$\begin{aligned} f(2) &= g(2) \\ a \cdot 2^3 + b \cdot 2^2 - 2 + 8 &= -5 \cdot 2 + 20 \\ 8a + 4b + 6 &= 10 \\ 8a + 4b &= 4 \qquad \text{divide by } 4 \\ 2a + b &= 1 \end{aligned}$$

Recall that the derivative of a linear function $l(x) = mx+b$ is the constant function $l'(x) = m$. Since $g(x) = -5x+20$ is linear, $g'(x) = -5$ for all x and so $g'(2) = -5$. To find $f'(2)$, we differentiate f and evaluate the derivative at $x = 2$.

$$\begin{aligned} f(x) &= ax^3 + bx^2 - x + 8 \\ f'(x) &= 3ax^2 + 2bx - 1 \\ f'(2) &= 3a \cdot 2^2 + 2b \cdot 2 - 1 = 12a + 4b - 1 \end{aligned}$$

$$\begin{aligned} f'(2) &= g'(2) \\ 12a + 4b - 1 &= -5 \\ 12a + 4b &= -4 \qquad \text{divide by } 4 \\ 3a + b &= -1 \end{aligned}$$

Now we have two equations in the variables a and b.

$$\begin{cases} 2a + b = 1 \\ 3a + b = -1 \end{cases}$$

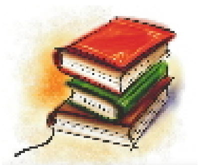
We will use elimination. We will multiply the first equation by -1 and add the two equations.

$$\begin{array}{rr} & -2a - b = -1 \\ + & 3a + b = -1 \\ \hline & a = -2 \end{array}$$

We substitute this in the first equation and solve for b

$$\begin{aligned} 2a + b &= 1 \\ 2(-2) + b &= 1 \\ -4 + b &= 1 \\ b &= 5 \end{aligned}$$

So the solution is: $a = -2$ and $b = 5$ and so $f(x) = -2x^3 + 5x^2 - x + 8$.



Sample Problems

1. Find an equation for the tangent line drawn to graph of $f(x) = x^4 - 2x^3 + 3$ at $x = -1$.
2. Let $f(x) = 16x + \dfrac{1}{x}$. Find the equation of all tangent lines drawn to the graph of f that are perpendicular to the line $x + 12y = -5$.
3. Find all values of p for which the line $y = x$ is a tangent line to the graph of $f(x) = px^2 + 6$.
4. Find an equation for all tangent lines drawn to the graph of $f(x) = \dfrac{1}{2}x^2 + 3x - 2$ from the point $P(-1, -9)$.

Practice Problems

1. Find an equation for the tangent line drawn to graph of f at the given value of x.

   a) $f(x) = 3x^4 + 2x - 7$ at $x = -1$

   b) $f(x) = x^2 - 12\sqrt{x}$ at $x = 4$

   c) $f(x) = \dfrac{1}{x^3} - \dfrac{1}{x}$ at $x = 1$

   d) $f(x) = -x^3 + 5x^2 + x - 8$ at $x = 2$

   e) $f(x) = 2x^3 + x^2 - 5x - 3$ at $x = -2$

   f) $f(x) = x^3 + 3x^2 + 3x - 12$ at $x = -1$

   g) $f(x) = 3x^4 + 6x^3 - 9x^2 - 8x + 2$ at $x = -2$

   h) $f(x) = 3x^4 + 6x^3 - 9x^2 - 8x + 2$ at $x = 1$

2. a) Let $f(x) = x^2 - 5x + 1$. Find the equation of all tangent lines drawn to the graph of f that are parallel to the line $x + y = 15$.

   b) Let $f(x) = x^3 - 8x + 6$. Find an equation for all tangent lines drawn to the graph of f that are perpendicular to the line $x + 4y = -12$.

   c) Let $f(x) = x - 3\sqrt{x}$. Find an equation for all tangent lines drawn to the graph of f that are perpendicular to the line $2x + y = 10$.

3. a) Find the values of a and b if we know that $y = -6x + 19$ is a tangent line drawn to the graph of $f(x) = ax^2 + bx + 3$ at $x = 2$.

   b) Find the values of m and n if we know that $y = 8x - 1$ is a tangent line drawn to the graph of $f(x) = mx^3 + nx^2 + x - 5$ at $x = -1$.

   c) Find the values of p and q if we know that $y = -\dfrac{1}{2}x + 7$ is a tangent line drawn to the graph of $f(x) = p\sqrt{x} + qx + 1$ at $x = 4$.

4. a) Find all values of p for which the line $y = 2x + 4$ is a tangent line to the graph of $f(x) = px^2 + 10$.

   b) Find all values of m for which the line $y = mx + \dfrac{1}{2}$ is a tangent line to the graph of $f(x) = \sqrt{x} - 2$.

   c) Find all values of A for which the line $y = 13x + 18$ is a tangent line to the graph of $f(x) = Ax^2 - 3x + 2$.

   d) Find all values of T for which the line $y = -2x + 12$ is a tangent line to the graph of $f(x) = T\sqrt{x} - 5x$.

5. In each case, find an equation for all tangent lines drawn to the graph of $f(x)$ given from the point given.

a) $y = x^2 - 6x + 1$ from $P(3, -9)$

b) $y = \frac{1}{2}x^2 - 6x + 30$ from $P(7, 8)$

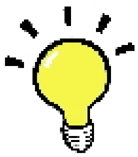
c) $y = x^2 - 2x - 8$ from $P(3, -14)$

d) $y = x^2 - 8x + 3$ from $P(0, -6)$

e) $f(x) = 7x - \frac{1}{2}x^2 - 20$ from $P(4, 8)$

f) $y = \frac{1}{2}x^2 + 3x - 11$ from $P(2, -5)$

g) $f(x) = \frac{1}{2}x^2 - x + 2$ from $P(3, -1)$



Answers

Sample Problems

1. $y = -10x - 4$ 2. $y = 12x + 4$ and $y = 12x - 4$ 3. $m = \frac{1}{24}$ 4. $y = -x - 10$ and $y = 5x - 4$

Practice Problems

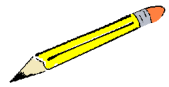
1. a) $y = -10x - 16$ b) $y = 5x - 28$ c) $y = -2x + 2$ d) $y = 9x - 12$
e) $y = 15x + 25$ f) $y = -13$ g) $y = 4x - 10$ h) $y = 4x - 10$

2. a) $y = -x - 3$ b) $y = 4x - 10$ and $y = 4x + 22$ c) $y = \frac{1}{2}x - \frac{9}{2}$

3. a) $a = -4$, $b = 10$ b) $m = 1$, $n = -2$ c) $p = 6, q = -2$

4. a) $p = \frac{1}{6}$ b) $m = \frac{1}{10}$ c) $A = -4$ d) $T = 12$

5. a) $y = 2x - 15$ and $y = -2x - 3$ b) $y = -2x + 22$ and $y = 4x - 20$ c) $y = -2x - 8$ and $y = 10x - 44$
d) $y = -2x - 6$ and $y = -14x - 6$ e) $y = -x + 12$ and $y = 7x - 20$ f) $y = 3x - 11, y = 7x - 19$
g) $y = -x + 2$ and $y = 5x - 16$

Sample Problems  Solutions

1. Find an equation for the tangent line drawn to graph of $f(x) = x^4 - 2x^3 + 3$ at $x = -1$.

   Solution: For the equation of a line, we need a point and the line's slope. The point of tangency $(-1, f(-1))$ is on the line. Since $f(-1) = (-1)^4 - 2(-1)^3 + 3 = 1 + 2 + 3 = 6$, the point $(-1, 6)$ is on the line. The derivative is the slope of the tangent line. So, the slope is the derivative, evaluated at $x = -1$, in short, $m = f'(-1)$.

   We differentiate $f(x)$.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 3 \\ f'(x) &= 4x^3 - 2\left(3x^2\right) + 0 = 4x^3 - 6x^2 \end{aligned}$$

   $f'(x) = 4x^3 - 6x^2$. We evaluate this at $x = -1$

$$f'(-1) = 4(-1)^3 - 6(-1)^2 = -4 - 6 = -10$$

   So the tangent line has slope -10 and passes through the point $(-1, 6)$. We can immediately write the point-slope form of the equation and transform it to the slope-intercept form.

$$\begin{aligned} y - 6 &= -10(x + 1) \\ y &= -10(x + 1) + 6 = -10x - 10 + 6 \\ y &= -10x - 4 \end{aligned}$$

2. Let $f(x) = 16x + \dfrac{1}{x}$. Find the equation of all tangent lines drawn to the graph of f that are perpendicular to the line $x + 12y = -5$.

   Solution: First we compute the slope of the line $x + 12y = -5$. For that, we need to bring the equation to the slope-intercept form by solving for y in terms of x.

$$\begin{aligned} x + 12y &= -5 \\ 12y &= -x - 5 \\ y &= \frac{-x - 5}{12} = -\frac{1}{12}x - \frac{5}{12} \end{aligned}$$

   Our line must be perpendicular to this line with slope $-\dfrac{1}{12}$. The slope of our line is therefore the negative reiprocal of $-\dfrac{1}{12}$, which is 12. So now the question is: where are the tangent lines drawn to f that have slopes 12? That is the same as looking for all values of x for which $f'(x) = 12$. We first differentiate $f(x)$.

$$\begin{aligned} f(x) &= 16x + \frac{1}{x} = 16x + x^{-1} \\ f'(x) &= 16 + (-1)x^{-2} = 16 - \frac{1}{x^2} \end{aligned}$$

   We now need to find all values of x for which $f'(x) = 12$.

$$\begin{aligned} f'(x) &= 12 & & \\ 16 - \frac{1}{x^2} &= 12 & & \text{add } \frac{1}{x^2} \\ 16 &= \frac{1}{x^2} + 12 & & \text{subtract } 12 \\ 4 &= \frac{1}{x^2} & & \text{mutiply by } x^2 \\ 4x^2 &= 1 & & \text{divide by } 4 \\ x^2 &= \frac{1}{4} \quad \Longrightarrow \quad x = \pm\frac{1}{2} & & \end{aligned}$$

We check: if $x=-\frac{1}{2}$, then $f'(x)=16-\left(\frac{1}{\left(-\frac{1}{2}\right)^2}\right)=16-\left(\frac{1}{\frac{1}{4}}\right)=16-4=12$. Similarly, if $x=\frac{1}{2}$, then $f'(x)=16-\left(\frac{1}{\left(\frac{1}{2}\right)^2}\right)=16-\left(\frac{1}{\frac{1}{4}}\right)=16-4=12$. For the tangent line, we find the points of tangency

$$f\left(\tfrac{1}{2}\right)=16\left(\tfrac{1}{2}\right)+\frac{1}{\left(\frac{1}{2}\right)}=8+2=10 \quad \text{and} \quad f\left(-\tfrac{1}{2}\right)=16\left(-\tfrac{1}{2}\right)+\frac{1}{\left(-\frac{1}{2}\right)}=-8-2=-10$$

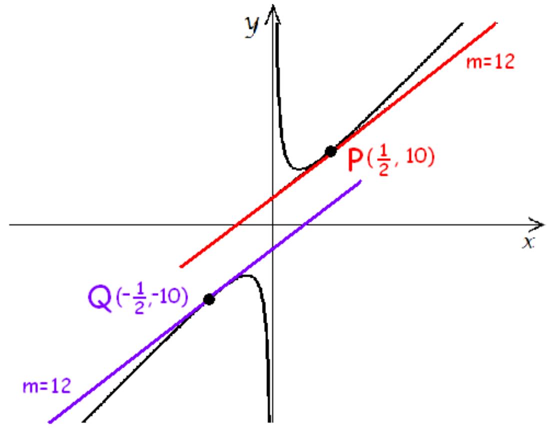
So one tangent line has slope $m=12$ and passes through $\left(\frac{1}{2},10\right)$. Its equation is

$$y-10=12\left(x-\frac{1}{2}\right) \quad \text{or} \quad y=12x+4$$

The other tangent line has slope $m=12$ and passes through $\left(-\frac{1}{2},-10\right)$ will have the equation

$$y+10=12\left(x+\frac{1}{2}\right) \quad \text{or} \quad y=12x-4$$

So the tangent lines we are looking for are $y=12x+4$ and $y=12x-4$.



3. Find all values of p for which the line $y=x$ is a tangent line to the graph of $f(x)=px^2+6$.

Solution: Let $P(a,f(a))$ be the point of tangency. Let us denote the tangent line $y=x$ by $g(x)$. Then, we can think of f and g as two functions that agree at $x=a$ to two levels:

$$f(a)=g(a) \quad \text{and} \quad f'(a)=g'(a)$$

These should give us two equations which we can solve for a and p.

$$\begin{aligned} f(a) &= g(a) \\ pa^2+6 &= a \end{aligned}$$

and for the derivatives: $f'(x)=2px$ and $g'(x)=1$.

$$\begin{aligned} f'(a) &= g'(a) \\ 2pa &= 1 \end{aligned}$$

We divide this last equation by 2 and obtain $pa=\frac{1}{2}$. We can substitute this into the other equation

$$\begin{aligned} pa^2+6 &= a \\ pa\cdot a+6 &= a \qquad pa=\frac{1}{2} \\ \frac{1}{2}a+6 &= a \\ 6 &= \frac{1}{2}a \\ 12 &= a \qquad \text{recall } pa=\frac{1}{2} \qquad 12p=\frac{1}{2} \qquad p=\frac{1}{24} \end{aligned}$$

So our function is $f(x) = \frac{1}{24}x^2 + 6$. We check:

$$f(x) = \frac{1}{24}x^2 + 6 \qquad f(12) = \frac{1}{24}(12)^2 + 6 = 6 + 6 = 12$$

$$f'(x) = \frac{1}{24}(2x) = \frac{1}{12}x \qquad f'(12) = \frac{1}{12}(12) = 1$$

So the tangent line drawn to f at $x = 12$ is

$$\begin{aligned} 1(x-12) &= y - 12 \\ x - 12 &= y - 12 \\ x &= y \end{aligned}$$

And so our solution is correct.

4. Find the equation of all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 + 3x - 2$ from the point $P(-1,-9)$.

Solution: This problem is more difficult because the point given, P is not on the graph of f. Let $Q(a, f(a))$ be the point of tangency. First, Q is on the graph of f

$$f(a) = \frac{1}{2}a^2 + 3a - 2$$

Second, the derivative of f, evaluated at $x = a$ is the slope of the tangent line.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x + 3 \qquad \Longrightarrow \qquad f'(a) = a + 3 \end{aligned}$$

This is also the slope of the line determined by $P(-1,-9)$ and $Q\left(a, \frac{1}{2}a^2 + 3a - 2\right)$.

$$m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(a) - (-9)}{a - (-1)} = \frac{\frac{1}{2}a^2 + 3a - 2 + 9}{a+1} = \frac{\frac{1}{2}a^2 + 3a + 7}{a+1}$$

The slope of the tangent line is the derivative evaluated at $x = a$.

$$\begin{aligned} m &= f'(a) \\ a + 3 &= \frac{\frac{1}{2}a^2 + 3a + 7}{a+1} && \text{multiply by } a+1 \\ (a+3)(a+1) &= \frac{1}{2}a^2 + 3a + 7 \\ a^2 + 4a + 3 &= \frac{1}{2}a^2 + 3a + 7 && \text{subtract 7} \\ a^2 + 4a - 4 &= \frac{1}{2}a^2 + 3a && \text{multiply by 2} \\ 2a^2 + 8a - 8 &= a^2 + 6a \\ a^2 + 2a - 8 &= 0 \\ (a+4)(a-2) &= 0 \qquad \Longrightarrow \qquad a_1 = -4, \ a_2 = 2 \end{aligned}$$

Case 1. If $a = -4$. Then $f(a) = \frac{1}{2}(-4)^2 + 3(-4) - 2 = -6$ and so the point of tangency is $(-4,-6)$. The slope of the tangent line is $f'(-4)$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x + 3 \\ f'(-4) &= -1 \end{aligned}$$

and so the tangent line is $-1(x+4) = y+6$ or $y = -x-10$. This line indeed contains $P(-1,-9)$.

Case 2. If $a = 2$. Then $f(a) = \frac{1}{2}(2)^2 + 3(2) - 2 = 6$ and so the point of tangency is $(2,6)$. The slope of the tangent line is $f'(2)$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 3x - 2 \\ f'(x) &= x+3 \\ f'(2) &= 5 \end{aligned}$$

and so the tangent line is $5(x-2) = y-6$ or $y = 5x-4$. This line also contains $P(-1,-9)$.

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