1. (Constant rule) If $f(x)=c$ where c is a constant, then $f'(x)=0$.
2. (Power rule) If $f(x)=x^n$, then $f'(x)=nx^{n-1}$
3. (Constant multiplier rule) For any differentiable function $f(x)$ and constant c, we define the function cf as $(cf)(x)=c\cdot f(x)$. Then $(cf)'(x)=c\cdot f'(x)$
4. (Sum rule) If f and g are differentiable functions, then so is $f+g$, and $(f+g)'(x)=f'(x)+g'(x)$.

Proofs of these theorems

1. (Constant rule) If $f(x)=c$ where c is a constant, then $f'(x)=0$.
Proof:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0 \ \blacksquare$$

2. (Power rule) If $f(x)=x^n$, then $f'(x)=nx^{n-1}$
We will not prove this general statement here, although it is true for every real number n. Instead, we will prove it for $n=2$, $n=3$, $n=-1$, and $n=\frac{1}{2}$

2A. Claim: If $f(x)=x^2$, then $f'(x)=2x$
Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}2x+h=2x \ \blacksquare\end{aligned}$$

2B. Claim: If $f(x)=x^3$, then $f'(x)=3x^2$
Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=\lim_{h\to 0}\frac{h\left(3x^2+3xh+h^2\right)}{h}=\lim_{h\to 0}3x^2+3xh+h^2=3x^2 \ \blacksquare\end{aligned}$$

2C. Claim: If $f(x)=\frac{1}{x}$, then $f'(x)=-\frac{1}{x^2}$
Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{x(x+h)}\cdot\frac{1}{h}\\ &= \lim_{h\to 0}\frac{x-x-h}{x(x+h)}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{-h}{x(x+h)}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2} \ \blacksquare\end{aligned}$$

2D. Claim: If $f(x)=\sqrt{x}$, then $f'(x)=\dfrac{1}{2\sqrt{x}}$

Proof:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\
&= \lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\ \blacksquare
\end{aligned}$$

3. (Constant multiplier rule) For any differentiable function $f(x)$ and constant c, we define the function cf as $(cf)(x)=c\cdot f(x)$. Then $(cf)'(x)=c\cdot f'(x)$

Proof:

$$\begin{aligned}
(cf)'(x) &= \lim_{h\to 0}\frac{(cf)(x+h)-(cf)(x)}{h}=\lim_{h\to 0}\frac{c\cdot f(x+h)-c\cdot f(x)}{h}=\lim_{h\to 0}\frac{c\left(f(x+h)-f(x)\right)}{h}\\
&= c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=c\cdot f'(x)\ \blacksquare
\end{aligned}$$

4. (Sum rule) If f and g are differentiable functions, then so is $f+g$, and $(f+g)'(x)=f'(x)+g'(x)$.

Proof:

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}=\lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}=\lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'(x)+g'(x)\ \blacksquare
\end{aligned}$$

