Recall that when we compose a function with its inverse, we obtain the identity function.

$$f\left(f^{-1}(x)\right) = x \quad \text{for all } x$$

Recall the chain rule: for differentiable functions f and g,

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

These two theorems are what we need to differentiate exponential functions. We will start with $g(x) = e^x$. We already know that $g(x) = e^x$ and $f(x) = \ln x$ are inverses of each other, i.e.

$$\ln(e^x) = x$$

We differentiate both sides of this equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} [\ln(e^x)]' &= (x)' \\ \frac{1}{e^x} \cdot \left[(e^x)'\right] &= 1 \qquad \text{we solve for } (e^x)' \\ (e^x)' &= e^x \end{aligned}$$

So, we have proved that $(e^x)' = e^x$. This is of course a very surprising and interesting property that we have only seen thus far with the constant zero function. It also quickly follows that the higher-order derivates are the same, i.e. if $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$, $f^{(4)}(x) = e^x$ and so on. It is also clear that the antiderivative of e^x is also the function itself, up to a constant added:

$$\int e^x dx = e^x + C$$

Example: Differentiate $h(x) = e^{7x}$.
Solution: We will use the chain rule. We define the outer function, $f(x) = e^x$ and the inner function, $g(x) = 7x$. Then

$$\begin{aligned} [f(g(x))]' &= f'(g(x)) \cdot g'(x) \text{ becomes} \\ \left[e^{7x}\right]' &= e^{7x} \cdot 7 = 7e^x \end{aligned}$$

Example: Differentiate $h(x) = \dfrac{1}{e^{4x}}$.
Solution: We can re-write $\dfrac{1}{e^{3x}}$ as e^{-3x} and apply the chain rule as before. We define the outer function, $f(x) = e^x$ and the inner function, $g(x) = -3x$. Then

$$\begin{aligned} [f(g(x))]' &= f'(g(x)) \cdot g'(x) \text{ becomes} \\ \left[e^{-3x}\right]' &= e^{-3x} \cdot (-3) = \frac{-3}{e^{3x}} \end{aligned}$$

What about other exponential functions such as $f(x) = 5^x$? We can also differentiate those using the chain rule. Recall first that

$$5^x = e^{\ln(5^x)}$$

This is true because e^x and $\ln x$ are inverses of each other, and so $e^{\ln A} = A$ for all positive A. We substitute $A = 5^x$ and obtain

$$e^{\ln(5^x)} = 5^x$$

Why does this help? If we re-write 5^x as $e^{\ln(5^x)}$, then we can use properties of logarithms, the chain rule, and that $(e^x)' = e^x$ to differentiate 5^x.

$$f(x) = 5^x = e^{\ln(5^x)} = e^{x \ln 5} = e^{(\ln 5)x}$$

Notice that $\ln 5$ is just a constant so the function $e^{(\ln 5)x}$ is very similar to a function like e^{7x} and can be easily differentiated by the chain rule. We define the outer function, $f(x) = e^x$ and the inner function, $g(x) = (\ln 5)x$. Then

$$\begin{aligned} [f(g(x))]' &= f'(g(x)) \cdot g'(x) \text{ becomes} \\ \left[e^{(\ln 5)x}\right]' &= e^{(\ln 5)x} \cdot (\ln 5) \end{aligned}$$

The first factor, $e^{(\ln 5)x}$ is still the same 5^x, and so we have that

$$(5^x)' = 5^x \cdot \ln 5$$