

## Sample Problems

1. Consider the straight line  $y = \frac{1}{2}x$  on the interval  $[0, 6]$ . Suppose we rotate the line around the  $x$ -axis. Find the volume of the object we obtain by
  - (a) first writing a Riemann sum estimating the volume
  - (b) turning the Riemann sum into an integral and evaluating it.
  - (c) Repeat the previous computations using the function  $f(x) = mx$  on the interval  $[0, h]$ , and  $r$  denoting the value of  $f(h) = mh$ .
2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is  $2 \frac{\text{N}}{\text{m}}$ .
  - (a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.
  - (b) Turn the Riemann sum into an integral and evaluate it to find the work.
3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that  $1 \text{ m}^3$  of water weighs 10 000 N, i.e. the 'density' of water is  $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$ . How much work does it require to pump all water out of the tank?
  - (a) Find a Riemann sum expressing the work.
  - (b) Turn the Riemann sum into an integral and evaluate it.
4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that  $1 \text{ m}^3$  of water weighs 10 000 N, i.e. the 'density' of water is  $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$ . How much work does it require to pump all water out of the tank?
  - (a) Find a Riemann sum expressing the work.
  - (b) Turn the Riemann sum into an integral and evaluate it.

## Sample Problems - Answers

$$1. \text{ a) } V_{\text{total}} = \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x \quad \text{or} \quad V_{\text{total}} = \sum_{i=1}^n \frac{\pi x_i^2}{4} \Delta x \quad \text{b) } 18\pi \quad \text{c) } \frac{\pi r^2 h}{3}$$

$$2. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n (370 - 2x_i) \Delta x \quad \text{b) } 18\,600 \text{ J}$$

$$3. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} (\delta\pi r^2) (10 - x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n (\delta\pi r^2) (10 - x_i) \Delta x$$

$$\text{b) } 4\,500\,000\pi \text{ J} \approx 1.413\,72 \times 10^7 \text{ J}$$

$$4. \text{ a) } W_{\text{total}} = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x \quad \text{or} \quad W_{\text{total}} = \sum_{i=1}^n \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x \quad \text{b) } 750\,000 \pi \text{ J} \approx 2.356\,2 \times 10^6 \text{ J}$$

## Sample Problems - Solutions

1. Consider the straight line  $y = \frac{1}{2}x$  on the interval  $[0, 6]$ . Suppose we rotate the line around the  $x$ -axis. Find the volume of the object we obtain by

a) first writing a Riemann sum estimating the volume

Solution: Let us partition the interval  $[0, 6]$  into  $n$  equal intervals by  $0 = x_0, x_1, x_2, \dots, x_n = 6$ , where the points are equally placed, and let  $\Delta x$  denote the length of a subinterval  $[x_i, x_{i+1}]$ . We will approximate the function  $f(x) = \frac{1}{2}x$  as constant  $f(x_i)$  over the entire interval  $[x_i, x_{i+1}]$ . Then the volume of a slice will be that of a cylinder with radius  $f(x_i)$  and height  $\Delta x$ . The volume of the cylindrical slice is

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi \left(\frac{x_i}{2}\right)^2 \Delta x = \frac{\pi x_i^2}{4} \Delta x$$

The total volume is

$$V_{\text{total}} = \sum_{i=0}^{n-1} V_i = \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x$$

b) turning the Riemann sum into an integral and evaluating it.

Solution: As  $n$  approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} V_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{\pi x_i^2}{4} \Delta x = \int_0^6 \left(\frac{\pi x^2}{4}\right) dx = \frac{\pi}{4} \int_0^6 x^2 dx = \frac{\pi}{4} \left(\frac{x^3}{3}\right) \Big|_0^6 \\ &= \frac{\pi}{4} \left(\frac{6^3}{3} - \frac{0^3}{3}\right) = \frac{\pi}{4} (72) = 18\pi \end{aligned}$$

- b) Repeat the previous computations using the function  $f(x) = mx$  on the interval  $[0, h]$ , and  $r$  denoting the value of  $f(h) = mh$ .

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi (mx_i)^2 \Delta x = \pi m^2 x_i^2 \Delta x$$

and thus the volume is

$$\begin{aligned} V_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} V_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \pi m^2 x_i^2 \Delta x = \int_0^h (\pi m^2 x^2) dx = \pi m^2 \int_0^h x^2 dx = \pi m^2 \left(\frac{x^3}{3}\right) \Big|_0^h \\ &= \pi m^2 \left(\frac{h^3}{3} - \frac{0^3}{3}\right) = \frac{\pi m^2 h^3}{3} = \frac{\pi (m^2 h^2) h}{3} = \frac{\pi r^2 h}{3} \end{aligned}$$

2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is  $2 \frac{\text{N}}{\text{m}}$ .

a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.

Solution: Let  $x$  represent the height of the bucket. Let us partition the interval  $[0, 60]$  into  $n$  equal intervals by  $0 = x_0, x_1, x_2, \dots, x_n = 60$ , where the points are equally placed, and let  $\Delta x$  denote the length of a subinterval  $[x_i, x_{i+1}]$ . We will approximate the height of the bucket to be  $x_i$  over the entire interval  $[x_i, x_{i+1}]$ . Then the force we have to exert while lifting the bucket from height  $x_i$  to  $x_{i+1}$  is

$$F_i = F_{\text{bucket}} + F_{\text{rope}} = 250 + 2(60 - x_i) = 370 - 2x_i$$

Then the work we have to invest when lifting the bucket from height  $x_i$  to  $x_{i+1}$  is

$$W_i = F_i \cdot s_i = (370 - 2x_i)(x_{i+1} - x_i) = (370 - 2x_i) \Delta x$$

The total work is then

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it to find the work.

Solution: As  $n$  approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x = \int_0^{60} (370 - 2x_i) dx = 370x - x^2 \Big|_0^{60} \\ &= (370(60) - (60)^2) - (370(0) - (0)^2) = 22\,200 - 3600 = 18\,600 \end{aligned}$$

(a) or, the same computation with the units, quite elegant,

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left( 370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x_i \right) \Delta x = \int_0^{60 \text{ m}} \left( 370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x \right) dx \\ &= \left( 370 \text{ N} x - x^2 \frac{\text{N}}{\text{m}} \right) \Big|_{0 \text{ m}}^{60 \text{ m}} \\ &= \left[ 370 \text{ N} (60 \text{ m}) - (60 \text{ m})^2 \frac{\text{N}}{\text{m}} \right] - \left[ 370 \text{ N} (0 \text{ m}) - (0 \text{ m})^2 \frac{\text{N}}{\text{m}} \right] \\ &= 22\,200 \text{ N m} - 3600 \text{ N m} = 18\,600 \text{ N m} \\ &= 18\,600 \text{ J} \end{aligned}$$

3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that  $1 \text{ m}^3$  of water weighs 10 000 N, i.e. the 'density' of water is  $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$ . How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let  $x$  represent the water level. Let us partition the interval  $[0, 10]$  into  $n$  equal intervals by  $0 = x_0, x_1, x_2, \dots, x_n = 10$ , where the points are equally placed, and let  $\Delta x$  denote the length of a subinterval  $[x_i, x_{i+1}]$ . We will approximate the water level to be  $x_i$  over the entire interval  $[x_i, x_{i+1}]$ . Then the force we have to exert when pumping out the water between heights  $x_i$  and  $x_{i+1}$  is the weight of a cylindrical 'slice' with volume

$$\begin{aligned} V_i &= \pi r^2 \Delta x \quad \text{the weight is then} \quad F_i = \delta V_i \\ F_i &= \delta \pi r^2 \Delta x \end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height  $x_i$  to 10 is

$$W_i = F_i \cdot s_i = (\delta \pi r^2 \Delta x) (10 - x_i) = (\delta \pi r^2) (10 - x_i) \Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As  $n$  approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (\delta\pi r^2) (10 - x_i) \Delta x = \int_0^{10} (\delta\pi r^2) (10 - x) dx = \delta\pi r^2 \int_0^{10} (10 - x) dx \\ &= 10\,000\pi (9) \int_0^{10} (10 - x) dx = 90\,000\pi \left( 10x - \frac{x^2}{2} \right) \Big|_0^{10} \\ &= 90\,000\pi \left[ \left( 10(10) - \frac{(10)^2}{2} \right) - \left( 10(0) - \frac{(0)^2}{2} \right) \right] \\ &= 90\,000\pi (50) = 4\,500\,000\pi \approx 1.413\,72 \times 10^7 \text{ J} \end{aligned}$$

4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m<sup>3</sup> of water weighs 10 000 N, i.e. the 'density' of water is  $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$ . How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let  $x$  represent the water level. Let us partition the interval  $[0, 10]$  into  $n$  equal intervals by  $0 = x_0, x_1, x_2, \dots, x_n = 10$ , where the points are equally placed, and let  $\Delta x$  denote the length of a subinterval  $[x_i, x_{i+1}]$ . We will approximate the water level to be  $x_i$  over the entire interval  $[x_i, x_{i+1}]$ . Then the force we have to exert when pumping out the water between heights  $x_i$  and  $x_{i+1}$  is the weight of a cylindrical 'slice' with volume  $\pi r_i^2 h$  where  $h = \Delta x$  and  $r_i$  is the radius at height  $x_i$ . Clearly  $\tan \alpha = \frac{r_i}{x_i} = \frac{3}{10}$  which gives us

that  $r_i = \frac{3}{10}x_i$

$$\begin{aligned} V_i &= \pi r_i^2 \Delta x = \pi \left( \frac{3}{10}x_i \right)^2 \Delta x = \frac{9\pi x_i^2}{100} \Delta x \text{ the weight is then } F_i = \delta V_i \\ F_i &= \frac{9\delta\pi x_i^2}{100} \Delta x \end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height  $x_i$  to 10 is

$$W_i = F_i \cdot s_i = \left( \frac{9\delta\pi x_i^2}{100} \Delta x \right) (10 - x_i) = \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As  $n$  approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned}W_{\text{total}} &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x = \int_0^{10} \frac{9\delta\pi x^2}{100} (10 - x) dx \\&= \frac{9\delta\pi}{100} \int_0^{10} x^2 (10 - x) dx = \frac{9(10\,000)\pi}{100} \int_0^{10} x^2 (10 - x) dx \\&= 900\pi \int_0^{10} (10x^2 - x^3) dx \\&= 900\pi \left( \frac{10x^3}{3} - \frac{x^4}{4} \right) \Big|_0^{10} \\&= 900\pi \left[ \left( \frac{10(10)^3}{3} - \frac{10^4}{4} \right) - \left( \frac{0^3}{3} - \frac{0^4}{4} \right) \right] \\&= 900\pi \left( \frac{10^4}{3} - \frac{10^4}{4} \right) = 900\pi \left( \frac{1}{12} \right) 10^4 \\&= 750\,000\pi\end{aligned}$$

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