Sample Problems

1. Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[1, 4]$.

   a) Compute the left Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.
   b) Compute the right Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.

2. Consider the function $f(x) = x^2$ on the interval $[0, 6]$.

   a) Compute the left Riemann sum for f on this interval with $n = 6$ subintervals.
   b) Compute the right Riemann sum for f on this interval with $n = 6$ subintervals.
   c) Compute the left Riemann sum for f on this interval with $n = 12$ subintervals.
   d) Compute the right Riemann sum for f on this interval with $n = 12$ subintervals.
   e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.
   f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.
   g) Compute the left Riemann sum for f on this interval with n subintervals.
   h) Compute the limit of the left Riemann sum for f on this interval with n intervals, as n approaches infinity.
   i) Compute the right Riemann sum for f on this interval with n subintervals.
   j) Compute the limit of the right Riemann sum for f on this interval with n intervals, as n approaches infinity.

Practice Problems

1. Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 4]$.

   a) Compute the left Riemann sum for f on this interval with $n = 4$ subintervals.
   b) Compute the right Riemann sum for f on this interval with $n = 4$ subintervals.
   c) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
   d) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.

2. Consider the function $f(x) = \ln(x+1)$ on the interval $[0, 10]$.

   a) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
   b) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.

3. Consider the function $f(x) = x^3$ on the interval $[0, 1]$.

   a) Compute the left Riemann sum for f on this interval with $n = 4$ subintervals.
   b) Compute the right Riemann sum for f on this interval with $n = 4$ subintervals.
   c) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
   d) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.
   e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.
   f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.

Answers - Sample Problems

1. a) $\dfrac{223}{140} \approx 1.592\,857$ b) $\dfrac{341}{280} \approx 1.217\,857$

2. a) 55 b) 91 c) $\dfrac{253}{4} = 63.25$ d) $\dfrac{325}{4} = 81.25$ e) $\dfrac{177\,309}{2500} = 70.924$

   f) $\dfrac{182\,709}{2500} = 73.084$ g) $\dfrac{36(n-1)(2n-1)}{n^2} = \dfrac{72n^2 - 108n + 36}{n^2}$ h) 72

   i) $\dfrac{36(2n^2+3n+1)}{n^2} = \dfrac{72n^2 + 108n + 36}{n^2}$ j) 72

Answers - Practice Problems

1. a) $1 + \sqrt{2} + \sqrt{3} \approx 4.146\,264\,37$ b) $3 + \sqrt{2} + \sqrt{3} \approx 6.146\,264\,37$

   c) $\dfrac{2\sqrt{10}}{25}\left(6 + \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8}\right) \approx 4.884\,075$

   d) $\dfrac{2\sqrt{10}}{25}\left(6 + \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{10}\right) \approx 5.684\,075$

2. a) $\ln(10!) = \ln 3628\,800 \approx 15.104\,412\,57$ b) $\ln(11!) = \ln 39\,916\,800 \approx 17.502\,307\,85$

3. a) $\dfrac{9}{64} = 0.140\,625$ b) $\dfrac{25}{64} = 0.390\,625$ c) $\dfrac{81}{400} = 0.202\,5$ d) $\dfrac{121}{400} = 0.302\,5$

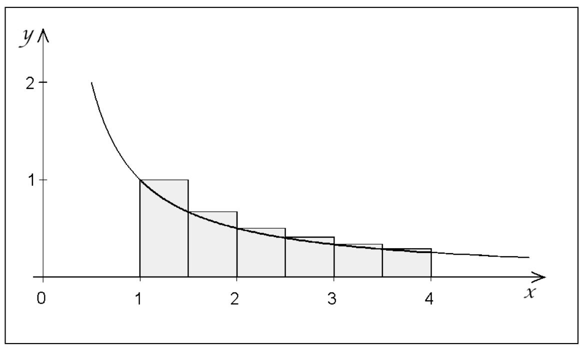
4. e) $\dfrac{9801}{40\,000} = 0.245\,025$ f) $\dfrac{10\,201}{40\,000} = 0.255\,025$

Sample Problems - Solutions

1. Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[1, 4]$.

   a) Compute the left Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.

   Solution: The interval $[1, 4]$ is 3 units long. The regular partition will contain intervals of length $\dfrac{3}{6} = \dfrac{1}{2}$. The partition consists of $\left\{1,\ 1\dfrac{1}{2}, 2,\ 2\dfrac{1}{2}, 3,\ 3\dfrac{1}{2},\ 4\right\}$ Notice that these are seven numbers. We usually start labeling with zero. In this case, these seven numbers are $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$. On each interval, we approximate the area under the graph by a rectangle as tall as the function value of the left endpoint of the interval. For example, on the first interval, we approximate the area under the graph using a rectangle with height $\dfrac{1}{1} = 1$. On the second interval, the height of the rectangle is $\dfrac{1}{1\frac{1}{2}} = \dfrac{2}{3}$.



The first rectangle has width $\dfrac{1}{2}$ and height 1. The area is $A_1 = \dfrac{1}{2} \cdot 1 = \dfrac{1}{2}$.

The second rectangle has width $\dfrac{1}{2}$ and height $\dfrac{1}{\left(\frac{3}{2}\right)} = \dfrac{2}{3}$. The area is $A_2 = \dfrac{1}{2} \cdot \dfrac{2}{3} = \dfrac{1}{3}$.

Let us notice that all rectangles have the same width of $\dfrac{1}{2}$. This is an advantage of a regular partition.

The third rectangle has height $\dfrac{1}{2}$. Its area is $A_3 = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}$.

The fourth rectangle has height $\dfrac{1}{\left(\frac{5}{2}\right)} = \dfrac{2}{5}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{2}{5} = \dfrac{1}{5}$

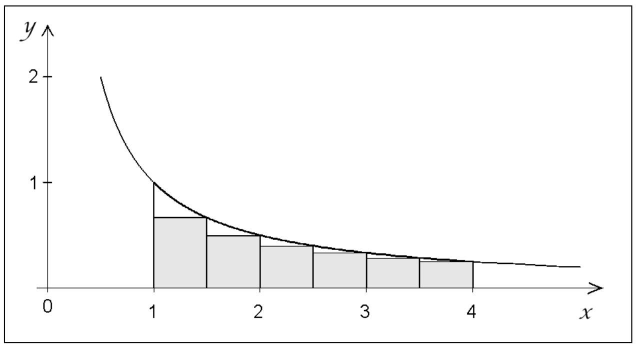
The fifth rectangle has height $\dfrac{1}{3}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{1}{3} = \dfrac{1}{6}$.

The sixth rectangle has height $\dfrac{1}{\left(\frac{7}{2}\right)} = \dfrac{2}{7}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{2}{7} = \dfrac{1}{7}$. In short, the left-hand approximation is

$$\begin{aligned} L_{f,n=6} &= \frac{1}{2}\cdot\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{1.5}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2.5}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3.5}=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{1.5}+\frac{1}{2}+\frac{1}{2.5}+\frac{1}{3}+\frac{1}{3.5}\right)\\ &= \frac{1}{2}\left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}\right)=\frac{223}{140}\approx 1.592\,857 \end{aligned}$$

We can clearly see from the picture that this approximation is an overestimation of the area.

b) Compute the right Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.



$$\begin{aligned} R &= \frac{1}{2}\cdot\frac{1}{1.5}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2.5}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3.5}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}\left(\frac{1}{1.5}+\frac{1}{2}+\frac{1}{2.5}+\frac{1}{3}+\frac{1}{3.5}+\frac{1}{4}\right)\\ &= \frac{1}{2}\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}+\frac{1}{4}\right)=\frac{341}{280}\approx 1.217\,857 \end{aligned}$$

We can clearly see from the picture that this approximation is an underestimation of the area. Thus we now know that the area under the graph is between those two values:

$$1.217\,857 < A < 1.592\,857$$

Note: We sometimes use summation notation when writing such expressions. Using summation notation, these Riemann sums are

$$L = \sum_{k=0}^{5}\frac{1}{2}\cdot\frac{1}{1+k\left(\frac{1}{2}\right)}=\frac{1}{2}\sum_{k=0}^{5}\frac{1}{1+k\left(\frac{1}{2}\right)}=\frac{1}{2}\sum_{k=0}^{5}\frac{1}{\frac{2+k}{2}}=\frac{1}{2}\sum_{k=0}^{5}\frac{2}{2+k}=\sum_{k=0}^{5}\frac{1}{2+k}$$

$$\text{and } R = \sum_{k=1}^{6}\frac{1}{2}\cdot\frac{1}{1+k\left(\frac{1}{2}\right)}=\sum_{k=1}^{6}\frac{1}{2+k}$$

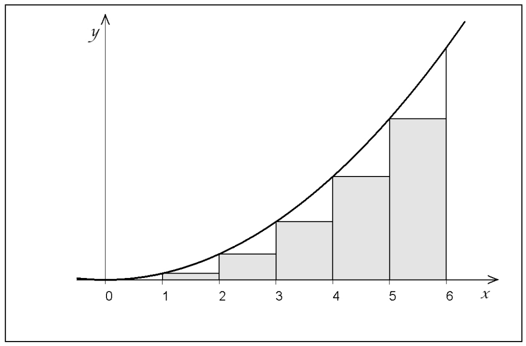
2. For this problem, we will need the following theorem: for all natural numbers n,

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Consider the function $f(x) = x^2$ on the interval $[0, 6]$.

a) Compute the left Riemann sum for f on this interval with $n = 6$ subintervals.

Solution: Each subinterval is of length 1, and so the partition is $\{0, 1, 2, 3, 4, 5, 6\}$.



The left-hand sum is

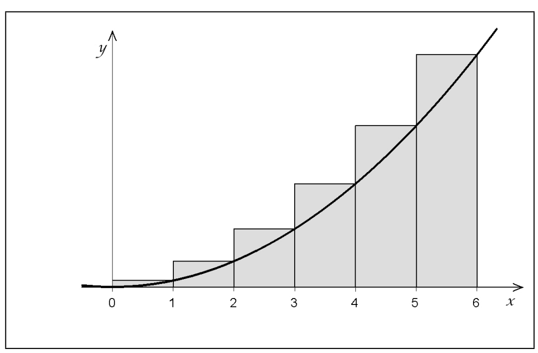
$$L_{f,n=6} = 1 \cdot 0^2 + 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 + 1 \cdot 4^2 + 1 \cdot 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

Using summation notation,

$$L_{f,n=6} = \sum_{k=0}^{5} 1 \cdot k^2 = \sum_{k=0}^{5} k^2 = 55$$

We can see on the picture that this Riemann sum underestimates the area.

b) Compute the right Riemann sum for f on this interval with $n = 6$ subintervals.



$$R_{f,n=6} = 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 + 1 \cdot 4^2 + 1 \cdot 5^2 + 1 \cdot 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

Using summation notation,

$$R_{f,n=6} = \sum_{k=1}^{6} 1 \cdot k^2 = \sum_{k=1}^{6} k^2 = 91$$

We can see on the picture that this Riemann sum underestimates the area. Thus, we have that

$$55 < A < 91$$

c) Compute the left Riemann sum for f on this interval with $n = 12$ subintervals.

Solution: Each subinterval will have length $\frac{6}{12} = \frac{1}{2}$. The partition is $\{0,\ 0.5,\ 1,\ 1.5\ ,2\ ,2.5,\ 3,\ 3.5,\ 4,\ 4.5,\ 5,\ 5.5,\ 6\}$. The left Riemann sum is

$$\begin{aligned} L_{f,n=12} &= \frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot 0.5^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot 1.5^2 + \frac{1}{2}\cdot 2^2 + \frac{1}{2}\cdot 2.5^2 + \frac{1}{2}\cdot 3^2 + \frac{1}{2}\cdot 3.5^2 + \frac{1}{2}\cdot 4^2 \\ &\quad + \frac{1}{2}\cdot 4.5^2 + \frac{1}{2}\cdot 5^2 + \frac{1}{2}\cdot 5.5^2 \end{aligned}$$

Although this looks like a lot of computation, it can be made quite simple using a bit of algebra and the theorem stated above. We first factor out $\frac{1}{2}$ and write the rest as fractions, with a common denominator of 2.

$$\begin{aligned} L_{f,n=12} &= \frac{1}{2}\left(0^2 + 0.5^2 + 1^2 + 1.5^2 + ... + 5.5^2\right) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + ... + \left(\frac{11}{2}\right)^2\right) \\ &= \frac{1}{2}\left(\frac{1^2}{4} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + ... + \frac{11^2}{4}\right) && \text{factor out } \frac{1}{4} \\ &= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2^2 + 3^2 + ... + 11^2\right) && \text{use theorem with } n = 11 \\ &= \frac{1}{8}\cdot\frac{11\cdot 12\cdot 23}{6} = \frac{253}{4} = 63.25 \end{aligned}$$

The same computation, using summation notation:

$$L_{f,n=12} = \sum_{k=0}^{11}\frac{1}{2}\cdot\left(\frac{1}{2}k\right)^2 = \frac{1}{2}\sum_{k=0}^{11}\frac{k^2}{4} = \frac{1}{8}\sum_{k=0}^{11}k^2 = \frac{1}{8}\frac{11\cdot 12\cdot 23}{6} = \frac{253}{4} = 63.25$$

d) Compute the right Riemann sum for f on this interval with $n = 12$ subintervals.

Solution: The difference between the left and right Riemann sums is just the first and the last rectangle.

$$\begin{aligned} R_{f,n=12} &= \frac{1}{2}\left(0.5^2 + 1^2 + ... + 5.5^2 + 6^2\right) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + ... + \left(\frac{12}{2}\right)^2\right) \\ &= \frac{1}{2}\left(\frac{1^2}{4} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + ... + \frac{12^2}{4}\right) && \text{factor out } \frac{1}{4} \\ &= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2^2 + 3^2 + ... + 12^2\right) && \text{use theorem with } n = 12 \\ &= \frac{1}{8}\cdot\frac{12\cdot 13\cdot 25}{6} = \frac{325}{4} = 81.25 \end{aligned}$$

Using summation notation,

$$R_{f,n=12} = \sum_{k=1}^{12}\frac{1}{2}\cdot\left(\frac{1}{2}k\right)^2 = \frac{1}{2}\sum_{k=1}^{12}\frac{k^2}{4} = \frac{1}{8}\sum_{k=1}^{12}k^2 = \frac{1}{8}\frac{12\cdot 13\cdot 25}{6} = \frac{325}{4} = 81.25$$

Because this function is increasing, all left sums underestimate the area and all right sums overestimate the area under the graph. Thus

$$63.25 < A < 81.25$$

e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.

Solution: Each subinterval is $\dfrac{6}{100}$ units long. The partition is

$$\left\{x_0 = 0,\ x_1 = \frac{6}{100},\ x_2 = \frac{12}{100},\ ....,x_k = \frac{6k}{100},...,x_{100} = \frac{600}{100} = 6\right\}$$

The left Riemann sum is

$$\begin{aligned}
L_{f,n=100} &= \frac{6}{100}\cdot 0^2 + \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2 + \frac{6}{100}\cdot\left(\frac{12}{100}\right)^2 + ... + \frac{6}{100}\cdot\left(\frac{6\cdot 99}{100}\right)^2 \\
&= \frac{6}{100}\left(0^2 + \left(\frac{6\cdot 1}{100}\right)^2 + \left(\frac{6\cdot 2}{100}\right)^2 + ... + \left(\frac{6\cdot 99}{100}\right)^2\right) \\
&= \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2 \left(0^2 + 1^2 + 2^2 + ... + 99^2\right) = \left(\frac{6}{100}\right)^3 \frac{99\cdot 100\cdot 199}{6} \\
&= \left(\frac{6}{100}\right)^2 (99\cdot 199) = \frac{177\,309}{2500} = 70.924
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
L_{f,n=100} &= \sum_{k=0}^{99} \frac{6}{100}\cdot\left(\frac{6}{100}k\right)^2 = \frac{6}{100}\sum_{k=0}^{99}\left(\frac{6}{100}\right)^2 k^2 = \left(\frac{6}{100}\right)^3 \sum_{k=0}^{99} k^2 = \left(\frac{6}{100}\right)^3 \frac{99\cdot 100\cdot 199}{6} \\
&= \left(\frac{6}{100}\right)^2 (99\cdot 199) = \frac{177\,309}{2500} = 70.924
\end{aligned}$$

f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.

The right Riemann sum is

$$\begin{aligned}
R_{f,n=100} &= \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2 + \frac{6}{100}\cdot\left(\frac{12}{100}\right)^2 + ... + \frac{6}{100}\cdot\left(\frac{6\cdot 100}{100}\right)^2 \\
&= \frac{6}{100}\left(\left(\frac{6\cdot 1}{100}\right)^2 + \left(\frac{6\cdot 2}{100}\right)^2 + ... + \left(\frac{6\cdot 100}{100}\right)^2\right) \\
&= \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2 \left(0^2 + 1^2 + 2^2 + ... + 100^2\right) = \left(\frac{6}{100}\right)^3 \frac{100\cdot 101\cdot 201}{6} \\
&= \left(\frac{6}{100}\right)^2 (101\cdot 201) = \frac{182\,709}{2500} = 73.083\,6
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
R_{f,n=100} &= \sum_{k=1}^{100} \frac{6}{100}\cdot\left(\frac{6}{100}k\right)^2 = \frac{6}{100}\sum_{k=1}^{100}\left(\frac{6}{100}\right)^2 k^2 = \left(\frac{6}{100}\right)^3 \sum_{k=1}^{100} k^2 = \left(\frac{6}{100}\right)^3 \frac{100\cdot 101\cdot 201}{6} \\
&= \left(\frac{6}{100}\right)^2 (100\cdot 201) = \frac{182\,709}{2500} = 73.084
\end{aligned}$$

Thus

$$70.924 < A < 73.084$$

g) Compute the left Riemann sum for f on this interval with n subintervals.

Solution: Each subinterval is $\frac{6}{n}$ units long. The numbers in the partition are

$$\left\{x_0 = 0,\ x_1 = \frac{6}{n},\ x_2 = 2\left(\frac{6}{n}\right),\ x_3 = 3\left(\frac{6}{n}\right),\ x_4 = 4\left(\frac{6}{n}\right), ...,\ x_n = n\left(\frac{6}{n}\right) = 6\right\}$$

The left Riemann sum is

$$\begin{aligned}
L_{f,n} &= \frac{6}{n}\cdot 0^2 + \frac{6}{n}\left(\frac{6}{n}\right)^2 + \frac{6}{n}\left(2\cdot\frac{6}{n}\right)^2 + \frac{6}{n}\left(3\cdot\frac{6}{n}\right)^2 + ... + \frac{6}{n}\left((n-1)\cdot\frac{6}{n}\right)^2 \\
&= \frac{6}{n}\left(0^2 + \left(\frac{6}{n}\right)^2 + \left(2\cdot\frac{6}{n}\right)^2 + \left(3\cdot\frac{6}{n}\right)^2 + ... + \left((n-1)\cdot\frac{6}{n}\right)^2\right) \\
&= \frac{6}{n}\left(1^2\cdot\left(\frac{6}{n}\right)^2 + 2^2\cdot\left(\frac{6}{n}\right)^2 + 3^2\cdot\left(\frac{6}{n}\right)^2 + ... + (n-1)^2\cdot\left(\frac{6}{n}\right)^2\right) \\
&= \frac{6}{n}\left(\frac{6}{n}\right)^2\left(1^2+2^2+3^2+....+(n-1)^2\right) = \left(\frac{6}{n}\right)^3\frac{(n-1)\left((n-1)+1\right)\left(2(n-1)+1\right)}{6} \\
&= \left(\frac{6}{n}\right)^3\frac{(n-1)\,n\,(2n-1)}{6} = \left(\frac{6}{n}\right)^2(n-1)(2n-1) = \frac{36(n-1)(2n-1)}{n^2} \\
&= \frac{36\left(2n^2-3n+1\right)}{n^2} = \frac{72n^2-108n+36}{n^2}
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
L_{f,n} &= \sum_{k=0}^{n-1}\frac{6}{n}\cdot\left(\frac{6}{n}k\right)^2 = \frac{6}{n}\sum_{k=0}^{n-1}\left(\frac{6}{n}\right)^2k^2 = \left(\frac{6}{n}\right)^3\sum_{k=0}^{n-1}k^2 = \left(\frac{6}{n}\right)^3\frac{(n-1)\left((n-1)+1\right)\left(2(n-1)+1\right)}{6} \\
&= \left(\frac{6}{n}\right)^3\frac{(n-1)\,n\,(2n-1)}{6} = \left(\frac{6}{n}\right)^2(n-1)(2n-1) = \frac{36(n-1)(2n-1)}{n^2} = \frac{72n^2-108n+36}{n^2}
\end{aligned}$$

h) Compute the limit of the left Riemann sum for f on this interval with n intervals, as n approaches infinity.

Solution:

$$\lim_{n\to\infty}\frac{72n^2-108n+36}{n^2} = \lim_{n\to\infty}\left(72-\frac{108}{n}+\frac{36}{n^2}\right) = 72$$

i) Compute the right Riemann sum for f on this interval with n subintervals.

Solution: The right Riemann sum is

$$\begin{aligned}
R_{f,n} &= \frac{6}{n}\left(\frac{6}{n}\right)^2 + \frac{6}{n}\left(2\cdot\frac{6}{n}\right)^2 + \frac{6}{n}\left(3\cdot\frac{6}{n}\right)^2 + ... + \frac{6}{n}\left(n\cdot\frac{6}{n}\right)^2 \\
&= \frac{6}{n}\left(\left(\frac{6}{n}\right)^2 + \left(2\cdot\frac{6}{n}\right)^2 + \left(3\cdot\frac{6}{n}\right)^2 + ... + \left(n\cdot\frac{6}{n}\right)^2\right) \\
&= \frac{6}{n}\left(1^2\cdot\left(\frac{6}{n}\right)^2 + 2^2\cdot\left(\frac{6}{n}\right)^2 + 3^2\cdot\left(\frac{6}{n}\right)^2 + ... + n^2\cdot\left(\frac{6}{n}\right)^2\right)
\end{aligned}$$

$$\begin{aligned} R_{f,n} &= \frac{6}{n}\left(\frac{6}{n}\right)^2\left(1^2+2^2+3^2+....+n^2\right) = \left(\frac{6}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \\ &= \left(\frac{6}{n}\right)^2 (n+1)(2n+1) = \frac{36(n+1)(2n+1)}{n^2} = \frac{36(2n^2+3n+1)}{n^2} = \frac{72n^2+108n+36}{n^2} \end{aligned}$$

Using summation notation,

$$\begin{aligned} R_{f,n} &= \sum_{k=1}^{n} \frac{6}{n}\cdot\left(\frac{6}{n}k\right)^2 = \frac{6}{n}\sum_{k=1}^{n}\left(\frac{6}{n}\right)^2 k^2 = \left(\frac{6}{n}\right)^3 \sum_{k=1}^{n} k^2 = \left(\frac{6}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \\ &= \left(\frac{6}{n}\right)^2 (n+1)(2n+1) = \frac{36(n+1)(2n+1)}{n^2} = \frac{72n^2+108n+36}{n^2} \end{aligned}$$

j) Compute the limit of the right Riemann sum for f on this interval with n intervals, as n approaches infinity.

Solution:

$$\lim_{n\to\infty} \frac{72n^2+108n+36}{n^2} = \lim_{n\to\infty}\left(72+\frac{108}{n}+\frac{36}{n^2}\right) = 72$$