Theorem: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1) Sum Rule $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2) Difference Rule $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$ for any $k \in \mathbb{R}$

Consequences:

1) Every non-zero constant multiple of a divergent series diverges.
2) If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge

Theorem (The Comparison Test) Let $\sum a_n$, $\sum b_n$, and $\sum c_n$ be series with non-negative terms. Suppose that for some integer $N$

$$a_n \leq b_n \leq c_n \quad \text{for all } n > N$$

1) If $\sum c_n$ converges, then $\sum b_n$ also converges.
2) If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Proof: Suppose that $\sum a_n$, $\sum b_n$, and $\sum c_n$ are series with $a_n \leq b_n \leq c_n$ for all $n > N$. If $c_n$ is convergent, then the sequence of all partial sums $\displaystyle\sum_{k=1}^{n} c_k$ is bounded from above by the sum $C$. Then the sequence of partial sums $\displaystyle\sum_{k=1}^{n} b_k$ is also bounded above because (assuming $n > N$)

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{N} b_k + \sum_{k=N+1}^{n} b_k \leq \sum_{k=1}^{N} b_k + \sum_{k=N+1}^{n} c_k \leq \sum_{k=1}^{N} b_k + C$$

Since $\{b_n\}$ is non-negative, the sequence $\displaystyle\sum_{k=1}^{n} b_k$ of partial sums is non-decreasing and so it is also convergent.

Now $D = \displaystyle\sum_{k=1}^{N} a_k$. If $\sum a_n$ diverges, then the sequence of partial sums $\displaystyle\sum_{k=1}^{n} a_k$ is not bounded from above. That means that for any $M \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that (again, assuming $m > N$)

$$\begin{aligned} M + D &\leq \sum_{k=1}^{m} a_k \leq \sum_{k=1}^{N} a_k + \sum_{k=N+1}^{n} a_k = D + \sum_{k=N+1}^{n} a_k \\ M + D &\leq D + \sum_{k=N+1}^{n} a_k \quad \Longrightarrow \quad M \leq \sum_{k=N+1}^{n} a_k \end{aligned}$$

Then

$$M \leq \sum_{k=N+1}^{n} a_k \leq \sum_{k=N+1}^{n} b_k \leq b_1 + b_2 + \ldots + b_N + \sum_{k=N+1}^{n} b_k = \sum_{k=1}^{n} b_k$$

 

and so the partial sums $\sum_{k=1}^{n} b_k$ are not bounded from above. Thus the series $\sum b_k$ is divergent.

Example 1.) $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$

Solution: if $n \geq 1$, then

$$\begin{aligned} n^2+2n &\geq n^2 \\ \frac{1}{n^2+2n} &\leq \frac{1}{n^2} \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$. You may recall that we have seen this series as a telescoping sum. To apply the comparison test takes much less work. On the other hand, this method does not give us what the sum is, only the fact that it exists.

Example 2.) $\sum_{n=0}^{\infty} \frac{n}{n^2+3}$

Solution: If $n$ is a very large number, the 3 added in the denominator becomes insignificant. Then the entire expression is very close to $\frac{1}{n}$, which is a divergent series. When proving divergence using the comparison test, we must find something divergent that is smaller than our expression. Often times this can be easily done using a non-zero constant multiplier. In this case, $\frac{n}{n^2+3}$ is slightly less than $\frac{1}{n}$. Consequently, we will not be able to say that $\frac{n}{n^2+3}$ is greater than $\frac{1}{n}$ but we can easily prove that it is greater than $\frac{1}{4} \cdot \frac{1}{n}$. Since for all $n \geq 1$

$$\begin{aligned} 1 &\leq n^2 \\ 3 &\leq 3n^2 \\ n^2+3 &\leq n^2+3n^2 = 4n^4 \\ \frac{1}{n^2+3} &\geq \frac{1}{4n^2} \qquad \text{multiply both sides by } n \geq 1 \\ \frac{n}{n^2+3} &\geq \frac{n}{4n^2} = \frac{1}{4n} \qquad \Longrightarrow \qquad \frac{n}{n^2+3} \geq \frac{1}{4n} \end{aligned}$$

The series $\sum_{n=1}^{\infty} \frac{1}{4n}$ is a constant times $\sum_{n=1}^{\infty} \frac{1}{n}$, it is divergent. Then, by the comparison test, $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ is also divergent.

Example 3.) $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

Solution:

$$\begin{aligned} 3^n+1 &\geq 3^n \\ \frac{1}{3^n+1} &\leq \frac{1}{3^n} \qquad \text{multiply by } 2^n > 0 \\ \frac{2^n}{3^n+1} &\leq \frac{2^n}{3^n} \end{aligned}$$

Since $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ is a geometric series with $|r| < 1$ and it converges. Then $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$ converges by the comparison test.

 

Example 4.) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{3^n-1}$

Solution: For all $n \geq 1$, we have that $3^n \geq 2$

$$\begin{aligned} 3^n &\geq 2 && \text{divide by } 2 \\ \frac{3^n}{2} &\geq 1 && \text{multiply by } -1 \\ -\frac{3^n}{2} &\leq -1 && \text{add } 3^n \end{aligned}$$

$$\begin{aligned} 3^n-\frac{3^n}{2} &\leq 3^n-1 && \\ \frac{3^n}{2} &\leq 3^n-1 && \text{take reciprocal of both sides} \\ \frac{2}{3^n} &\geq \frac{1}{3^n-1} && \text{multiply by } 2^n \\ \frac{2\cdot 2^n}{3^n} &\geq \frac{2^n}{3^n-1} && \end{aligned}$$

Since $\displaystyle\sum_{n=0}^{\infty}\frac{2\cdot 2^n}{3^n}=2\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ is convergent, and for all $n\geq 1$, $\dfrac{2^n}{3^n-1}\leq\dfrac{2\cdot 2^n}{3^n}$, the series $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{3^n-1}$ converges by the comparison test

Example 5.) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^{3/2}}$

Solution: We will apply the comparison test using the inequality $\ln n \leq n^{1/4}$ that is true for large $n$. Let us first prove this statement. Consider the function $f(x)=x^{1/4}-\ln x$. We will prove that if $x\geq e^{16}$, then $f(x)$ is positive for all $x$.
Consider first $f\left(e^{16}\right)$.

$$f\left(e^{16}\right)=\left(e^{16}\right)^{1/4}-\ln\left(e^{16}\right)=e^4-16\geq 2.5^4-16=23.0625>0$$

Thus $f\left(e^{16}\right)>0$.
Now consider $f'(x)=\dfrac{1}{4x^{3/4}}-\dfrac{1}{x}$

$$\begin{aligned} \frac{1}{4x^{3/4}}-\frac{1}{x} &> 0 && \\ \frac{1}{4x^{3/4}} &> \frac{1}{x} && \text{multiply by } 4x \\ \sqrt[4]{x} &> 4 && \\ x &> 4^4 && \\ x &> 256 && \end{aligned}$$

Note that $e^{16}>256$. Consider now the function $f(x)=\sqrt[4]{x}-\ln x$ on domain $\left(e^{16},\infty\right)$. On this domain, $f'$ is positive and thus $f$ is increasing. Since $f\left(e^{16}\right)$ is positive, the function is positive on its entire domain. Consequently, if $n>e^{16}$, then $f(n)=n^{1/4}-\ln n>0$ and so $n^{1/4}>\ln n$. Thus

$$\frac{\ln n}{n^{3/2}}\leq\frac{n^{1/4}}{n^{3/2}}=n^{1/4-3/2}=n^{-5/4}=\frac{1}{n^{5/4}}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/4}}$ converges by the integral test, so does $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^{3/2}}$ by the comparison test.

# Practice Problems

Determine which of the following series converge and which diverge.

Please note that most of these problems can be solved using several different methods.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+1}{3^n}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$
5. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-2n}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{4^n+1}$
7. $\displaystyle\sum_{n=1}^{\infty} n^{-5/4}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^2+1}$
10. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{e^n}$
13. $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech} n$
14. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
16. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$
17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(1+\frac{1}{n}\right)^n}$

# Answers - Practice Problems

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the integral test or by grouping of terms

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges to 1, it is a telescoping sum $\displaystyle\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ $\qquad s_n=1-\frac{1}{n}$

   also by the comparison test $\displaystyle\frac{1}{n^2+n}\leq\frac{1}{n^2}$ which is convergent

3. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+1}{3^n}$ converges; it is a sum of two geometric series

   Solution: This is a sum of two geometric sequences. Not only converges, we can actually figure out the sum.
   $$\sum_{n=0}^{\infty}\frac{2^n+1}{3^n}=\sum_{n=0}^{\infty}\frac{2^n}{3^n}+\frac{1}{3^n}=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=3+\frac{3}{2}=\frac{9}{2}$$

4. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ converges by the integral test or by the comparison test: $\displaystyle\frac{1}{n^2+1}\leq\frac{1}{n^2}$ and $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2}$ converges

5. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^2-2n}$ converges by the comparison test

   Solution: For all $n\geq 4$,

   $$\begin{aligned} n &\geq 4 && \text{multiply by } n\\ n^2 &\geq 4n && \text{divide by 2}\\ \frac{n^2}{2} &\geq 2n && \text{subtract } 2n \end{aligned}$$

   $$\begin{aligned} \frac{n^2}{2}-2n &\geq 0 && \text{add } \frac{n^2}{2}\\ n^2-2n &\geq \frac{n^2}{2} && \text{take reciprocal of both sides}\\ \frac{1}{n^2-2n} &\leq \frac{2}{n^2} \end{aligned}$$

   Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so does $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2}$. Since for all $n\geq 4$, $\displaystyle\frac{1}{n^2-2n}\leq\frac{2}{n^2}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-2n}$ converges by the comparison test.

6. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{4^n+1}$ diverges by the comparison test and the constant multiple rule:
   $$\frac{5^n}{4^n+1}\geq\frac{5^n}{4^n+4^n}=\frac{5^n}{2\cdot 4^n}=\frac{1}{2}\cdot\left(\frac{5}{4}\right)^x$$

7. $\displaystyle\sum_{n=1}^{\infty}n^{-5/4}$ converges by the integral test

8. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{n}}$ diverges because it fails the $n$th term test

9. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^2+1}$ converges by the comparison or integral test

10. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by the comparison test

If $n \geq 3$, then $\ln n \geq 1$ and so $\dfrac{\ln n}{n} \geq \dfrac{1}{n}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges as well.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ converges by the comparison test

For all $n \in \mathbb{N}$, $\tan^{-1} n < \dfrac{\pi}{2}$ and so $\dfrac{8\tan^{-1} n}{1+n^2} \leq \dfrac{8\left(\frac{\pi}{2}\right)}{1+n^2} = 4\pi \cdot \dfrac{1}{1+n^2}$ and $\displaystyle\sum_{n=1}^{\infty} 4\pi \cdot \frac{1}{1+n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{1+n^2}$ is convergent by the integral or comparison test.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{e^n}$ converges because it is a geometric series with $|r| < 1$

13. $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech} n$ converges by the comparison test: $\operatorname{sech} n = \dfrac{2}{e^n + e^{-n}} \leq \dfrac{2}{e^n}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges by the comparison test

diverges by the comparison test: $\dfrac{n+2}{n^2-n} \geq \dfrac{1}{n}$ and $\dfrac{1}{n}$ is divergent

method 1: $\dfrac{n+2}{n^2-n} \geq \dfrac{n+2}{n^2} \geq \dfrac{n}{n^2} = \dfrac{1}{n}$

method 2: We want to prove that $\dfrac{n+2}{n^2-n} \geq \dfrac{1}{n}$ Same as $\dfrac{n+2}{n^2-n} - \dfrac{1}{n} \geq 0$

and $\dfrac{n+2}{n^2-n} - \dfrac{1}{n} = \dfrac{3}{n(n-1)}$ is clearly positive for all $n \geq 2$.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the integral test

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$ converges by the comparison test

$\dfrac{(n-1)!}{(n+2)!} = \dfrac{(n-1)!}{(n-1)!n(n+1)(n+2)} = \dfrac{1}{n(n+1)(n+2)} \leq \dfrac{1}{(n)(n)(n)} = \dfrac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(1+\frac{1}{n}\right)^n}$ diverges by the comparison test: $\dfrac{n}{\left(1+\frac{1}{n}\right)^n} > \dfrac{n}{e}$ or by the $n$th term test

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