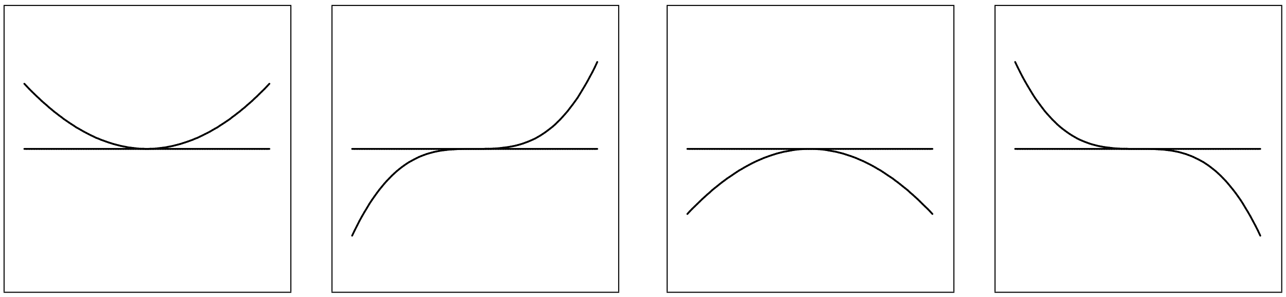
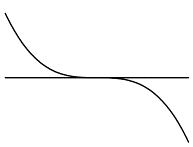
Consider a function $f$ that is twice differentiable on an open interval containing $c$. Suppose further that $f'(c) = 0$. Let us look at the function $f'$ first. Since $f$ is twice differentiable, $f'$ is differentiable. Differentiable functions have just a few ways in which they take a zero value.



$f'(x)$ $f'(x)$ $f'(x)$ $f'(x)$

Case 1. Suppose that $f'(c) = 0$ and $f''(c)$ is negative.

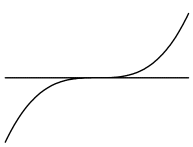


$f'(x)$

That indicates that $f'$ is strictly decreasing on an interval containing $c$. Taking a zero value while decreasing means that $f'$ changes sign from positive to negative. That indicates that $f$ has a relative maximum at $c$.

**Theorem:** If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a relative maximum at $c$.

Case 2. Suppose that $f'(c) = 0$ and $f''(c)$ is positive.



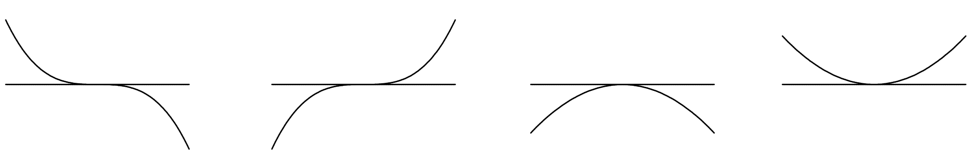
$f'(x)$

That indicates that $f'$ is strictly increasing on an interval containing $c$. Taking a zero value while increasing means that $f'$ changes sign from negative to positive. That indicates that $f$ has a relative minimum at $c$.

**Theorem:** If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a relative minimum at $c$.

Case 3. Suppose that $f'(c) = 0$ and $f''(c) = 0$. Consider $f(x) = x^8$ and $g(x) = x^9$ near $x = 0$.
$f$ has a relative minimum at $x = 0$ and $g$ has neither a maximum nor a minimum at $x = 0$, yet

$f'(0) = 0$, $f''(0) = 0$ and $g'(0) = 0$, $g''(0) = 0$



$f'(x)$ $f'(x)$ $f'(x)$ $f'(x)$

Furthermore, the higher order derivatives of $f$ and $g$ will be zero for quite a while. Therefore, the second derivative in this case did not distinguish between maximums and minimums.

**Theorem:** If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test did not yield for any useful result.

In such cases, we need to apply other methods to tell maximums, minimums, (or neither) apart.

**Example 1.** Suppose that $f(x) = \sin\left(\frac{1}{x}\right)$. Prove that $f$ has a relative maximum at $x = \frac{2}{\pi}$.

**Solution:** We differentiate $f$ and then $f'$: $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$ and $f''(x) = \frac{2}{x^3}\cos\frac{1}{x} - \frac{1}{x^4}\sin\frac{1}{x}$

Now we compute $f'\left(\frac{2}{\pi}\right)$ and $f'\left(\frac{2}{\pi}\right)$

$$f'\left(\frac{2}{\pi}\right) = -\frac{1}{\left(\frac{2}{\pi}\right)^2}\cos\left(\frac{1}{\left(\frac{2}{\pi}\right)}\right) = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}\right) = 0$$

and $f''\left(\frac{2}{\pi}\right) = 2\left(\frac{\pi}{2}\right)^3\cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^4\sin\left(\frac{\pi}{2}\right) = -\frac{1}{16}\pi^4$

So we have that $f'\left(\frac{2}{\pi}\right) = 0$ and $f''\left(\frac{2}{\pi}\right)$ is negative. Therefore, by the second derivative test, $f$ has a relative maximum at $\frac{2}{\pi}$.

The second derivative test worked in this case. Furthermore, we needed it, because it is difficult to sort out how $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$ changes sign at $x = \frac{2}{\pi}$.

