# Equations



Solve each of the following equations. Make sure to check your solutions.

1. Linear Equations

(a) $3(x-7)-2(5x-1)=3x+1$ $-2$

Solution: First we get rid of the parentheses by applying the distributive law.

$$\begin{aligned} 3(x-7)-2(5x-1) &= 3x+1 && \text{multiply out parentheses} \\ 3x-21-10x+2 &= 3x+1 && \text{combine like terms} \\ -7x-19 &= 3x+1 && \text{add } 7x \text{ to both sides} \\ -19 &= 10x+1 && \text{subtract 1 from both sides} \\ -20 &= 10x && \text{divide both sides by 10} \\ -2 &= x \end{aligned}$$

We check: if $x=-2$, then

$$\begin{aligned} \text{LHS} &= 3(-2-7)-2(5(-2)-1)=3(-9)-2(-10-1) \\ &= -27-2(-11)=-27-(-22)=-5 \\ \text{RHS} &= 3(-2)+1=-6+1=-5 \end{aligned}$$

Thus our solution, $x=-2$ is correct.

(b) $3(x-2)-2(5x-4)=2-x$ $0$

Solution: First we get rid of the parentheses by applying the distributive law.

$$\begin{aligned} 3(x-2)-2(5x-4) &= 2-x && \text{multiply out parentheses} \\ 3x-6-10x+8 &= 2-x && \text{combine like terms} \\ -7x+2 &= 2-x && \text{add } 7x \text{ to both sides} \\ 2 &= 6x+2 && \text{subtract 2 from both sides} \\ 0 &= 6x && \text{divide both sides by 6} \\ 0 &= x \end{aligned}$$

We check: if $x=0$, then

$$\begin{aligned} \text{LHS} &= 3(0-2)-2(5(0)-4)=3(-2)-2(0-4) \\ &= -6-2(-4)=-6-(-8)=2 \\ \text{RHS} &= 2-0=2 \end{aligned}$$

Thus our solution, $x=0$ is correct.

(c) $3(x-2)-2(5x-4)=11-7x$ no solution

Solution: First we get rid of the parentheses by applying the distributive law.

$$\begin{aligned} 3(x-2)-2(5x-4) &= 11-7x && \text{multiply out parentheses} \\ 3x-6-10x+8 &= 11-7x && \text{combine like terms} \\ -7x+2 &= 11-7x && \text{add } 7x \text{ to both sides} \\ 2 &= 11 \end{aligned}$$

When $x$ completely disappears from the equation, we are left with an unconditional statement. In this case, the statement, $2=11$ is unconditionally false. There is no value for $x$ that can ever make $2=11$ happen. Thus this equation is what we call a contradiction, and it has no solution.

(d) $\dfrac{3x-1}{5}-\dfrac{5-x}{2}=4x+6$ $-3$

Solution: (One trick with these equations is not to distibute too soon.) We first express the right hand side as a fraction as well.

$$\begin{aligned}
\frac{3x-1}{5}-\frac{5-x}{2} &= 4x+6 \\
\frac{3x-1}{5}-\frac{5-x}{2} &= \frac{4x+6}{1} && \text{bring fractions to the common denominator} \\
\frac{2(3x-1)}{10}-\frac{5(5-x)}{10} &= \frac{10(4x+6)}{10} && \text{multiply both sides by 10} \\
2(3x-1)-5(5-x) &= 10(4x+6) && \text{MOST IMPORTANT LINE TO HAVE!}
\end{aligned}$$

Now the problem is reduced to one similar to the previously solved equations.

$$\begin{aligned}
2(3x-1)-5(5-x) &= 10(4x+6) && \text{multiply out parentheses} \\
6x-2-25+5x &= 40x+60 && \text{combine like terms} \\
11x-27 &= 40x+60 && \text{subtract } 11x \\
-27 &= 29x+60 && \text{subtract 60} \\
-87 &= 29x && \text{divide by 29} \\
-3 &= x
\end{aligned}$$

We check: if $x=-3$, then

$$\begin{aligned}
\text{LHS} &= \frac{3(-3)-1}{5}-\frac{5-(-3)}{2}=\frac{-9-1}{5}-\frac{8}{2}=\frac{-10}{5}-\frac{8}{2}=-2-4=-6 \\
\text{RHS} &= 4(-3)+6=-12+6=-6
\end{aligned}$$

Thus our solution, $x=-3$, is correct.

(e) $\dfrac{x+4}{2}+\dfrac{5-2x}{3}=\dfrac{22-x}{6}$ identity; all numbers are solution

Solution:

$$\begin{aligned}
\frac{x+4}{2}+\frac{5-2x}{3} &= \frac{22-x}{6} && \text{bring fractions to the common denominator} \\
\frac{3(x+4)}{6}+\frac{2(5-2x)}{6} &= \frac{22-x}{6} && \text{multiply both sides by 6} \\
3(x+4)+2(5-2x) &= 22-x && \text{multiply out parentheses} \\
3x+12+10-4x &= 22-x && \text{combine like terms} \\
-x+22 &= 22-x && \text{rearrange right-hand side} \\
-x+22 &= -x+22
\end{aligned}$$

We are left with an equation where the left-hand side and the right-hand side are literally the same. It is easy to see that the two sides will have the same value for any number we substitute into $x$. An equation like this is called an identity, and all numbers are solution.

Note: we do not have to stop here, we can continue to simplify the equation as

$$\begin{aligned}
-x+22 &= -x+22 && \text{add } x \text{ to both sides} \\
22 &= 22
\end{aligned}$$

and now we have an unconditionally true statement, indicating that any value of $x$ will work with the equation.

2. Quadratic or higher degree equations

(a) $x^2 = 25$ $5, -5$

Solution: Since this is a quadratic equation, we will reduce one side to zero and factor the other side. Then we apply the zero property.

$$\begin{aligned} x^2 &= 25 && \text{subtract 25} \\ x^2 - 25 &= 0 && \text{notice that } 25 = 5^2 \\ x^2 - 5^2 &= 0 && \text{factor via the difference of squares theorem} \\ (x+5)(x-5) &= 0 \end{aligned}$$

Now we apply the zero property.

$$\begin{aligned} x + 5 &= 0 \quad \text{or} \quad x - 5 = 0 \\ x &= -5 \quad \text{or} \quad x = 5 \end{aligned}$$

We check: if $x = 5$, then

$$\text{LHS} = 5^2 = 25 = \text{RHS}$$

and if $x = -5$, then

$$\text{LHS} = (-5)^2 = 25 = \text{RHS}$$

Thus both values, 5 and $-5$ are correct.

(b) $6x^2 = 7x + 3$ $-\frac{1}{3}, \frac{3}{2}$

Solution: Since the equation is quadratic, we reduce one side to zero, then we factor the other side, and finally apply the zero property.

$$\begin{aligned} 6x^2 &= 7x + 3 \\ 6x^2 - 7x - 3 &= 0 \end{aligned}$$

We factor by grouping. First we conduct the "$pq-$game". We first write

$$\begin{aligned} pq &= \\ p + q &= \end{aligned}$$

Then we complete the right-hand sides: the sum $p + q$ must be the linear coefficient (i.e. the number in front of $x$, with its sign), $-7$. The product $pq$ must be the product of the quadratic and constant coefficients (i.e. multiply the signed number in front of $x^2$ by the signed number at the end) is $6(-3) = -18$. Thus we need to find $p$ and $q$ satisfying

$$\begin{aligned} pq &= -18 \quad \text{and} \\ p + q &= -7 \end{aligned}$$

There is only a few ways to express 18 as a product of two numbers:

$$1 \cdot 18 \quad \text{or} \quad 2 \cdot 9 \quad \text{or} \quad 3 \cdot 6$$

Since the product of $p$ and $q$ must be negative, it is clear that one of the numbers must be positive, the other one must be negative. If we look at the sum $p + q = -7$, we see that the larger in the pair should carry the negative sign to make the sum negative. This greatly reduces the possible candidates for $p$ and $q$, since we only need to consider

$$\begin{array}{lll} 1 & \text{with} & -18 \quad \text{or} \\ 2 & \text{with} & -9 \quad \text{or} \\ 3 & \text{with} & -6 \quad \text{or} \end{array}$$

Clearly only 2 and $-9$ work with the sum $p+q$.
We will use these numbers to factor by grouping. We express the linear term as the sum of two terms

$$-7x = 2x - 9x$$

and this will enable us to factor the polynomial by grouping.

$$\begin{aligned} 6x^2 - 7x - 3 &= 0 \\ \underbrace{6x^2 + 2x}\underbrace{-9x - 3} &= 0 \\ 2x\,(3x+1) - 3\,(3x+1) &= 0 \\ (2x-3)\,(3x+1) &= 0 \end{aligned}$$

We now apply the zero property. Either

$$\begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

or

$$\begin{aligned} 3x + 1 &= 0 \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

We check: if $x = \dfrac{3}{2}$, then

$$\begin{aligned} \text{LHS} &= 6\left(\frac{3}{2}\right)^2 = 6\left(\frac{9}{4}\right) = \frac{6}{1}\cdot\frac{9}{4} = \frac{54}{4} = \frac{27}{2} \\ \text{RHS} &= 7\left(\frac{3}{2}\right) + 3 = \frac{7}{1}\cdot\frac{3}{2} + 3 = \frac{21}{2} + 3 = \frac{21}{2} + \frac{6}{2} = \frac{27}{2} \end{aligned}$$

and if $x = -\dfrac{1}{3}$, then

$$\begin{aligned} \text{LHS} &= 6\left(-\frac{1}{3}\right)^2 = 6\left(\frac{1}{9}\right) = \frac{6}{1}\cdot\frac{1}{9} = \frac{6}{9} = \frac{2}{3} \\ \text{RHS} &= 7\left(-\frac{1}{3}\right) + 3 = \frac{7}{1}\cdot\frac{-1}{3} + 3 = \frac{-7}{3} + 3 = \frac{-7}{3} + \frac{9}{3} = \frac{2}{3} \end{aligned}$$

Thus both numbers, $\dfrac{3}{2}$ and $-\dfrac{1}{3}$ are correct.

(c) $30x^3+4x^4=154x^2$ $\frac{7}{2}, 0, -11$

Solution: Since this equation is of a degree higher than 1, we reduce one side to zero, factor the other side, and then apply the zero property.

$$\begin{aligned}30x^3+4x^4 &= 154x^2 && \text{subtract } 154x^2 \text{ and rearrange terms by degree}\\ 4x^4+30x^3-154x^2 &= 0 && \text{factor out the GCF (greatest common factor)}\\ 2x^2\left(2x^2+15x-77\right) &= 0 && \text{divide both sides of the equation by 2}\\ x^2\left(2x^2+15x-77\right) &= 0\end{aligned}$$

We will now factor by grouping. First we conduct the "$pq-$game".

$$\begin{aligned}pq &= -154\\ p+q &= 15\end{aligned}$$

The possible ways to express 154 as a product are: $154=2\times7\times11$

$$1\cdot154 \quad \text{or} \quad 2\cdot77 \quad \text{or} \quad 7\cdot22 \quad \text{or} \quad 11\cdot14$$

Since the product $pq$ is negative, exactly one of the numbers must carry a negative sign. Since $p+q$ is positive, the negative sign must be carried by the smaller one in the pair. So we only need to consider

$$-1 \text{ with } 154 \qquad \text{or} \qquad -2 \text{ with } 77 \qquad \text{or} \qquad -7 \text{ with } 22 \qquad \text{or} \qquad -11 \text{ with } 14$$

and clearly only $-7$ and 22 work with the sum. Now we proceed to factor.

$$\begin{aligned}x^2\left(2x^2+15x-77\right) &= 0\\ x^2\left(\underbrace{2x^2+22x}\underbrace{-7x-77}\right) &= 0\\ x^2\left(2x\left(x+11\right)-7\left(x+11\right)\right) &= 0\\ x^2\left(2x-7\right)\left(x+11\right) &= 0\end{aligned}$$

We now apply the zero property. The product $x\cdot x\cdot(2x-7)\cdot(x+11)$ can only be zero if either

$$x=0 \quad \text{or} \quad 2x-7=0 \quad \text{or} \quad x+11=0$$

We solve these linear equations for $x$ and obtain the three solutions, 0, $\frac{7}{2}$, and $-11$. We check: if $x=0$, then

$$\begin{aligned}\text{LHS} &= 30\left(0\right)^3+4\left(0\right)^4=0\\ \text{RHS} &= 154\left(0\right)^2=0\end{aligned}$$

If $x=\frac{7}{2}$, then

$$\begin{aligned}\text{LHS} &= 30\left(\frac{7}{2}\right)^3+4\left(\frac{7}{2}\right)^4=30\left(\frac{343}{8}\right)+4\left(\frac{2401}{16}\right)\\ &= \frac{10290}{8}+\frac{9604}{16}=\frac{5145}{4}+\frac{2401}{4}=\frac{7546}{4}=\frac{3773}{2}\\ \text{RHS} &= 154\left(\frac{7}{2}\right)^2=154\left(\frac{49}{4}\right)=\frac{7546}{4}=\frac{3773}{2}\end{aligned}$$

If $x=-11$, then

$$\begin{aligned}\text{LHS} &= 30(-11)^3+4(-11)^4 = 30(-1331)+4(14641)\\ &= -39930+58564 = 18634\\ \text{RHS} &= 154(-11)^2 = 154(121) = 18634\end{aligned}$$

and so all three values of $x$, $0, \frac{7}{2}$, and $-11$ are correct.

3. Absolute value equations.

(a) $|2x-7|+3=4$ $3, 4$
Solution: we first have to isolate the expression within the absolute value signs.

$$\begin{aligned}|2x-7|+3 &= 4 \quad \text{subtract 3}\\ |2x-7| &= 1\end{aligned}$$

Now we trade in one absolute value equation for two linear equations. The expression $2x-7$ has absolute value 1. This is only possible in two ways:

$$\text{either } 2x-7=1 \quad \text{or} \quad 2x-7=-1$$

We solve both these linear equations for $x$.

$$\begin{aligned}2x-7 &= 1 \quad \text{add 7}\\ 2x &= 8 \quad \text{divide by 2}\\ x &= 4\end{aligned}$$

and

$$\begin{aligned}2x-7 &= -1 \quad \text{add 7}\\ 2x &= 6 \quad \text{divide by 2}\\ x &= 3\end{aligned}$$

We check. If $x=3$, then

$$\begin{aligned}\text{LHS} &= |2(3)-7|+3 = |6-7|+3 = |-1|+3 = 1+3 = 4\\ \text{RHS} &= 4\end{aligned}$$

and if $x=4$, then

$$\begin{aligned}\text{LHS} &= |2(4)-7|+3 = |8-7|+3 = |1|+3 = 1+3 = 4\\ \text{RHS} &= 4\end{aligned}$$

Thus both values, 3 and 4 are solutions of the equation.

(b) $\left|3x-\frac{1}{2}\right|+4=1$ no solution
Solution: we first have to isolate the expression within the absolute value signs.

$$\begin{aligned}\left|3x-\frac{1}{2}\right|+4 &= 1 \quad \text{subtract 4}\\ \left|3x-\frac{1}{2}\right| &= -3\end{aligned}$$

Since no absolute value can be negative, this equation has no solution.

(c) $2\left|3x-7\right|-4=-4$ $\frac{7}{3}$
Solution: we first have to isolate the expression within the absolute value signs.

$$\begin{aligned} 2\left|3x-7\right|-4 &= -4 \quad \text{add } 4 \\ 2\left|3x-7\right| &= 0 \quad \text{divide by } 2 \\ \left|3x-7\right| &= 0 \end{aligned}$$

The only way $3x-7$ can have absolute value zero if its value is zero. Thus we have to solve

$$\begin{aligned} 3x-7 &= 0 \quad \text{add } 7 \\ 3x &= 7 \\ x &= \frac{7}{3} \end{aligned}$$

We check. If $x=\frac{7}{3}$, then

$$\begin{aligned} \text{LHS} &= 2\left|3\left(\frac{7}{3}\right)-7\right|-4=2\left|7-7\right|-4=2\left|0\right|-4=2(0)-4=0-4=-4 \\ \text{RHS} &= -4 \end{aligned}$$

Thus our solution $\frac{7}{3}$ is correct.