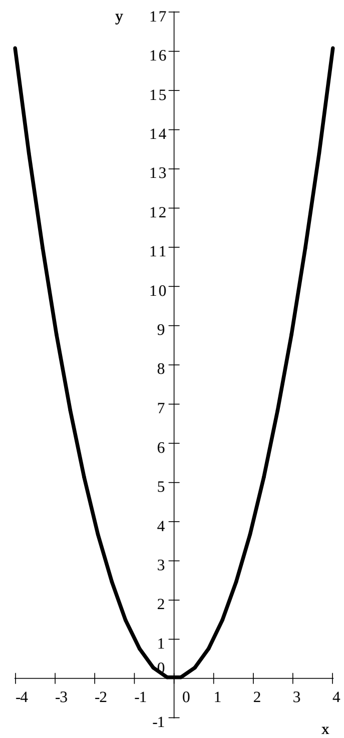
The Parabola

The simplest parabola is defined by the equation $y = x^2$. We obtain points on the parabola by choosing x-values and then computing the y-coordinates using the equation of the parabola.

For example, if $x = -3$, then $y = (-3)^2 = 9$, and so this computation yielded the point $(-3, 9)$ on the parabola. Computing the points with x-coordinates $-3, -2, -1, 0, 1, 2$, and 3 gives us the graph:



The graph of every quadratic expression is a parabola, and therefore shares basic characteristics with the parabola $y = x^2$.

Definition 1 *The **vertex** of a parabola is the lowest or highest point before rounding back upon itself. Every parabola has a vertex.*

Definition 2 *The y-**intercept** of a graph is the point where the curve intersects the y-axis.*

Definition 3 *The x-**intercept(s)** of a graph is (are) the point(s) where the curve intersects the x-axis.*

When graphing a straight line, we only needed to find any two points on the graph, and they told us all about the line. It is trickier with the parabola. To graph a parabola, we have to find the vertex and the four points around the vertex, thereby capturing the essential part of the graph. Every quadratic expression has several algebraic forms, and each of them are useful to find different types of points on the parabola.

Example 4 *Graph the parabola defined by the equation $y = x^2 + 2x - 3$. Label the coordinates of the vertex, the y-intercept, and the x-intercepts.*

This form, $y = x^2 + 2x - 3$ is called the **polynomial form**. It is very easy to find the y-intercept using this form. Just plug in $x = 0$ and find y:

$$\begin{aligned} y &= 0^2 + 2(0) - 3 \\ y &= -3 \end{aligned}$$

Thus we obtained the point $(0, -3)$, which is the y-intercept.

Let us now complete the square. We start with the polynomial form:

$$y = x^2 + 2x - 3$$

Next we take the coefficient of the second term, with its sign, and divide it by 2. We get $\frac{+2}{2} = +1$. We place an x in front of this and square: $(x+1)^2$. We FOIL the expression to obtain $(x+1)^2 = x^2 + 2x + 1$. Now we know that we need to smuggle in 1 to complete the square. So we re-write our original expression as

$$y = x^2 + 2x + 1 - 1 - 3$$

and then the **complete square form** is

$$y = (x+1)^2 - 4.$$

The complete square form tells us where the vertex is if we know how to read it. All we need to do is to think of the vertex as the lowest point on the graph. Play the following question-and-answer game.

Q. What is the lowest value that the expression $(x+1)^2 - 4$ can achieve?

A. The first part, $(x+1)^2$ is a square, so its lowest possible value is 0. Then the lowest value of the entire expression $(x+1)^2 - 4$ is -4.

Q. What does x has to be equal to for this lowest value to be achieved?

A. The square part, $(x+1)^2$ has to be zero. So $(x+1)^2 = 0$ means $x + 1 = 0$ giving us $x = -1$.

And so the coordinates of the vertex are $(-1, -4)$.

After we have completed the square, we can easily factor the expression using the difference of squares theorem. We start with the complete square form:

$$y = (x+1)^2 - 4$$

Then we re-write 4 as the square of 2:

$$y = (x+1)^2 - (2)^2$$

So now we can apply the difference of squares theorem.

$$y = (x+1+2)(x+1-2)$$

Now we simplify the factors.

$$y = (x+3)(x-1)$$

And this the **factored form** of the quadratic expression.

The factored form tells us where the x-intercepts of the parabola are. The x-intercepts are points of whose y-coordinate is 0. We simply need to solve the equation $y = 0$. Remember, we just computed that $y = (x+3)(x-1)$.

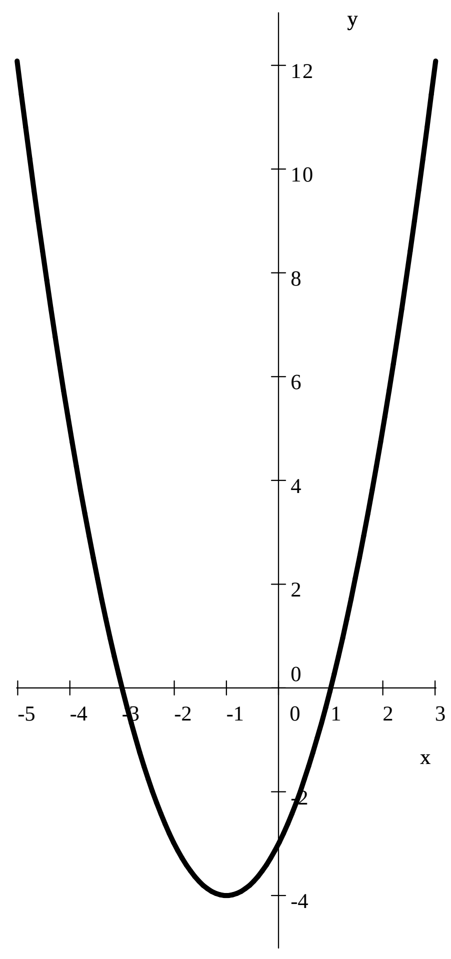
$$\begin{aligned} y &= 0 \\ (x+3)(x-1) &= 0 \\ x &= -3 \text{ or } x = 1 \end{aligned}$$

This computation gave us the points $(-3, 0)$ and $(1, 0)$.

To graph a parabola, we first need to know the x-coordinate of the vertex. In our case, it is $x = -1$. Thus we need to compute the y-coordinates for the points with x-coordinates $-3, -2, -2, 0$, and 1. We plug this numbers into x in the formula $y = x^2 + 2x - 3$.

$$\begin{aligned} \text{If } x &= -3, \text{ then } y = (-3)^2 + 2(-3) - 3 = 0, \text{ so we have } (-3, 0). \\ \text{If } x &= -2, \text{ then } y = (-2)^2 + 2(-2) - 3 = -3, \text{ so we have } (-2, -3). \\ \text{If } x &= -1, \text{ then } y = (-1)^2 + 2(-1) - 3 = -4, \text{ so we have } (-1, -4). \\ \text{If } x &= 0, \text{ then } y = (0)^2 + 2(0) - 3 = -3, \text{ so we have } (0, -3). \\ \text{If } x &= 1, \text{ then } y = (1)^2 + 2(1) - 3 = 0, \text{ so we have } (1, 0). \end{aligned}$$

Now we are ready to graph:



We the label the other required points: the vertex is $(-1,-4)$, the y-intercept is $(0,-3)$, and the x-intercepts are $(-3,0)$ and $(1,0)$.

Example 5 *Graph the parabola defined by the equation $y = x^2 - 6x - 7$. Label the coordinates of the vertex, the y-intercept, and the x-intercepts.*

We are given the polynomial form. This immediately gives us the y-intercept $(0,-7)$. Further computation will yield for the vertex and the x-intercepts.

$$\begin{aligned}
y &= x^2 - 6x - 7 \qquad \text{thus the } y\text{-intercept is } (0,-7) \\
y &= x^2 - 6x - 7 \qquad\qquad (x-3)^2 = x^2 - 6x + 9 \\
y &= x^2 - 6x + 9 - 9 - 7 \\
y &= (x-3)^2 - 16 \qquad \text{thus the vertex is } (3,-16) \\
y &= (x-3)^2 - 4^2 \\
y &= (x-3+4)(x-3-4) \\
y &= (x+1)(x-7) \quad \text{thus the } x\text{-intercepts are } (-1,0) \text{ and } (7,0)
\end{aligned}$$

Because of the x-coordinate of the vertex is 3, we need to find the y-coordinates for $x = 1, 2, 3, 4$, and 5. Some of these were already found. We use the polynomial form, $y = x^2 - 6x - 7$.

$$\begin{aligned}
\text{If } x &= 1, \text{ then } y = (1)^2 - 6(1) - 7 = -12 \qquad \text{Found } (1,-12) \\
\text{If } x &= 2, \text{ then } y = (2)^2 - 6(2) - 7 = -15 \qquad \text{Found } (2,-15) \\
\text{If } x &= 3, \text{ we already know } (3,-16) \text{ is the vertex} \\
\text{If } x &= 4, \text{ then } y = (4)^2 - 6(4) - 7 = -15 \qquad \text{Found } (4,-15) \\
\text{If } x &= 5, \text{ then } y = (5)^2 - 6(5) - 7 = -12 \qquad \text{Found } (5,-12)
\end{aligned}$$

In addition we have the x-intercepts: $(-1,0)$ and $(7,0)$. This is plenty of information to plot the graph: