Deriving The Quadratic Formula

Let $ax^2 + bx + c = 0$ be a quadratic equation, where $a \neq 0$.

1. We factor out the leading coefficient, a.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0 \end{aligned}$$

2. We divide the second coefficient, $\frac{b}{a}$ by 2:

$$\frac{b}{a} \div 2 = \frac{b}{a} \cdot \frac{1}{2} = \frac{b}{2a}$$

3. Thus the complete square we need is

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \\ &= x^2 + \frac{b}{2a}x + \frac{b}{2a}x + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \end{aligned}$$

4. Now we smuggle in the missing term for the complete square, $\frac{b^2}{4a^2}$

$$\begin{aligned} a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \end{aligned}$$

5. We bring the fractions after the complete square to the common denominator, and factor out -1

$$\begin{aligned} a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c \cdot 4a}{a \cdot 4a}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} - \frac{-4ac}{4a^2}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) &= 0 \end{aligned}$$

6. We prepare to factor via the Difference of Squares Theorem

$$a\left(\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2\right)=0$$

7. The second expression simplifies since

$$\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\sqrt{b^2-4ac}}{2a}$$

8. And now we factor via the Difference of Squares Theorem

$$\begin{aligned} a\left(\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2\right) &= 0 \\ a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right) &= 0 \end{aligned}$$

9. And now we apply the special zero property:

$$\begin{aligned} x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} &= 0 \\ x+\frac{\sqrt{b^2-4ac}}{2a} &= -\frac{b}{2a} \\ x &= -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}$$

or

$$\begin{aligned} x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a} &= 0 \\ x-\frac{\sqrt{b^2-4ac}}{2a} &= -\frac{b}{2a} \\ x &= -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b+\sqrt{b^2-4ac}}{2a} \end{aligned}$$

So the general two solutions of a quadratic equation

$$ax^2+bx+c=0 \quad \text{where} \quad a\neq 0$$

are x_1 and x_2 and they can be computed by the formula

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The expression b^2-4ac (the expression under the square-root sign) is called the discriminant. When the discriminant is positive, the equation has two different solutions. When the discriminant is zero, the equation has one solution. When the discriminant is negative, the equation has no solution.