Equations Containing Radical Expressions – SOLUTIONS

1. $\sqrt{3x-2} = x$

   Solution:

$$\begin{aligned} \sqrt{3x-2} &= x && \text{square} \\ 3x-2 &= x^2 && \text{reduce one side to zero} \\ 0 &= x^2-3x+2 && \text{factor} \\ 0 &= (x-2)(x-1) \\ x_1 &= 2 \quad \text{and} \quad x_2 = 1 \end{aligned}$$

   We check: if $x=2$, then

$$\begin{aligned} \text{LHS} &= \sqrt{3(2)-2} = \sqrt{4} = 2 \\ \text{RHS} &= 2 \end{aligned}$$

   and if $x=1$, then

$$\begin{aligned} \text{LHS} &= \sqrt{3(1)-2} = \sqrt{1} = 1 \\ \text{RHS} &= 1 \end{aligned}$$

   Thus the solution set is: $\{1, 2\}$

2. $\sqrt[3]{3x-6} = 3$

   Solution:

$$\begin{aligned} \sqrt[3]{3x-6} &= 3 && \text{raise both sides to the third power} \\ 3x-6 &= 27 && \text{add 6} \\ 3x &= 33 && \text{divide by 3} \\ x &= 11 \end{aligned}$$

   We check: if $x=11$, then

$$\begin{aligned} \text{LHS} &= \sqrt[3]{3(11)-6} = \sqrt[3]{27} = 3 \\ \text{RHS} &= 3 \end{aligned}$$

   , Solution: 11

3. $10 + \sqrt{4x-7} = 7$

   Solution:

$$\begin{aligned} 10 + \sqrt{4x-7} &= 7 && \text{subtract 10} \\ \sqrt{4x-7} &= -3 \end{aligned}$$

   Since no square root is negative, there is no solution.

4. $5+\sqrt{x+15}=x$

Solution:

$$\begin{aligned} 5+\sqrt{x+15} &= x && \text{subtract 5} \\ \sqrt{x+15} &= x-5 && \text{square both sides} \\ x+15 &= (x-5)^2 && \text{FOIL right hand side} \\ x+15 &= x^2-10x+25 && \text{reduce one side to zero} \\ 0 &= x^2-11x+10 && \text{factor} \\ 0 &= (x-1)(x-10) && \text{Special Zero Property} \\ x_1 &= 1 \quad \text{and } x_2=10 \end{aligned}$$

We check: If $x=1$, then

$$\begin{aligned} \text{LHS} &= 5+\sqrt{1+15}=5+\sqrt{16}=5+4=9 \\ \text{RHS} &= 1 \\ \text{RHS} &\neq \text{LHS} \end{aligned}$$

Thus $x=1$ is NOT a solution.

If $x=10$, then

$$\begin{aligned} \text{LHS} &= 5+\sqrt{10+15}=5+\sqrt{25}=5+5=10 \\ \text{RHS} &= 10 \\ \text{RHS} &= \text{LHS} \end{aligned}$$

Thus $x=10$ is the only solution.

5. $2\sqrt{x-1}=x-4$

Solution:

$$\begin{aligned} 2\sqrt{x-1} &= x-4 && \text{square both sides} \\ 4(x-1) &= (x-4)^2 && \text{FOIL, distribute} \\ 4x-4 &= x^2-8x+16 && \text{reduce one side to zero} \\ 0 &= x^2-12x+20 && \text{factor} \\ 0 &= (x-2)(x-10) \\ x_1 &= 2 \quad \text{and} \quad x_2=10 \end{aligned}$$

We check: If $x=2$, then

$$\begin{aligned} \text{LHS} &= 2\sqrt{2-1}=2\sqrt{1}=2(1)=2 \\ \text{RHS} &= 2-4=-2 \\ \text{RHS} &\neq \text{LHS} \end{aligned}$$

Thus $x=2$ is NOT a solution.

If $x = 10$, then

$$\begin{aligned} \text{LHS} &= 2\sqrt{10-1} = 2\sqrt{9} = 2(3) = 6 \\ \text{RHS} &= 10 - 4 = 6 \\ \text{RHS} &= \text{LHS} \end{aligned}$$

Thus $x = 10$ is the only solution.

6. $2\sqrt{x+4} = 1 + \sqrt{2x+9}$

Solution:

$$\begin{aligned} 2\sqrt{x+4} &= 1 + \sqrt{2x+9} && \text{square both sides} \\ \left(2\sqrt{x+4}\right)^2 &= \left(1 + \sqrt{2x+9}\right)^2 \\ \left(2\sqrt{x+4}\right)\left(2\sqrt{x+4}\right) &= \left(1 + \sqrt{2x+9}\right)\left(1 + \sqrt{2x+9}\right) \\ 4(x+4) &= 1 + \sqrt{2x+9} + \sqrt{2x+9} + 2x + 9 && \text{combine like terms} \\ 4x + 16 &= 2x + 10 + 2\sqrt{2x+9} && \text{subtract } 2x \\ 2x + 16 &= 10 + 2\sqrt{2x+9} && \text{subtract 10} \\ 2x + 6 &= 2\sqrt{2x+9} \\ 2(x+3) &= 2\sqrt{2x+9} && \text{divide by 2} \\ x + 3 &= \sqrt{2x+9} && \text{square both sides} \\ (x+3)^2 &= 2x + 9 \\ x^2 + 6x + 9 &= 2x + 9 && \text{reduce one side to zero} \\ x^2 + 4x &= 0 && \text{factor} \\ x(x+4) &= 0 \\ x_1 &= 0 \quad \text{and} \quad x_2 = -4 \end{aligned}$$

We check: If $x = 0$, then

$$\begin{aligned} \text{LHS} &= 2\sqrt{0+4} = 2\sqrt{4} = 2(2) = 4 \\ \text{RHS} &= 1 + \sqrt{2(0)+9} = 1 + 3 = 4 \end{aligned}$$

Thus $x = 0$ is indeed a solution.

If $x = -4$, then

$$\begin{aligned} \text{LHS} &= 2\sqrt{(-4)+4} = 2\sqrt{0} = 2(0) = 0 \\ \text{RHS} &= 1 + \sqrt{2(-4)+9} = 1 + \sqrt{-8+9} = 1 + \sqrt{1} = 1 + 1 = 2 \\ \text{RHS} &\neq \text{LHS} \end{aligned}$$

Thus $x = -4$ is NOT a solution. The only solution is $x = 0$.

7. $5\sqrt{x} + 1 = 3\sqrt{x} + 17$

Solution:

$$\begin{aligned} 5\sqrt{x}+1 &= 3\sqrt{x}+17 && \text{subtract } 3\sqrt{x} \\ 2\sqrt{x}+1 &= 17 && \text{subtract 1} \\ 2\sqrt{x} &= 16 && \text{divide by 2} \\ \sqrt{x} &= 8 && \text{square both sides} \\ x &= 64 \end{aligned}$$

We check: If $x = 64$, then

$$\begin{aligned} \text{LHS} &= 5\sqrt{64}+1 = 5(8)+1 = 41 \\ \text{RHS} &= 3\sqrt{64}+17 = 3(8)+17 = 24+17 = 41 \end{aligned}$$

Thus $x = 64$ is indeed a solution.

8. $\sqrt{2x+5}+5 = x$ (9.6 Example 3)

Solution:

$$\begin{aligned} \sqrt{2x+5}+5 &= x && \text{subtract 5} \\ \sqrt{2x+5} &= x-5 && \text{square} \\ 2x+5 &= x^2-10x+25 && \text{reduce one side to zero} \\ 0 &= x^2-12x+20 && \text{factor} \\ 0 &= (x-2)(x-10) \\ x_1 &= 2 \quad \text{and} \quad x_2 = 10 \end{aligned}$$

We check: If $x = 2$, then

$$\begin{aligned} \text{LHS} &= \sqrt{2(2)+5}+5 = \sqrt{4+5}+5 = \sqrt{9}+5 = 3+5 = 8 \\ \text{RHS} &= 2 \\ \text{RHS} &\neq \text{LHS} \end{aligned}$$

Thus $x = 2$ is NOT a solution.

If $x = 10$, then

$$\begin{aligned} \text{LHS} &= \sqrt{2(10)+5}+5 = \sqrt{20+5}+5 = \sqrt{25}+5 = 5+5 = 10 \\ \text{RHS} &= 10 \\ \text{RHS} &= \text{LHS} \end{aligned}$$

Thus $x = 10$ is the only solution.

9. $\sqrt{2x+5}-\sqrt{x-1} = \sqrt{x+2}$

Solution:

$$\begin{aligned}
\sqrt{2x+5}-\sqrt{x-1} &= \sqrt{x+2} && \text{add } \sqrt{x-1} \\
\sqrt{2x+5} &= \sqrt{x+2}+\sqrt{x-1} && \text{square} \\
\left(\sqrt{2x+5}\right)^2 &= \left(\sqrt{x+2}+\sqrt{x-1}\right)^2 \\
2x+5 &= \left(\sqrt{x+2}+\sqrt{x-1}\right)\left(\sqrt{x+2}+\sqrt{x-1}\right) \\
2x+5 &= \underbrace{\sqrt{x+2}\sqrt{x+2}}_{\text{F}}+\underbrace{\sqrt{x+2}\sqrt{x-1}}_{\text{O}}+\underbrace{\sqrt{x-1}\sqrt{x+2}}_{\text{I}}+\underbrace{\sqrt{x-1}\sqrt{x-1}}_{\text{L}} \\
2x+5 &= x+2+2\sqrt{x-1}\sqrt{x+2}+x-1 \\
2x+5 &= 2x+1+2\sqrt{x-1}\sqrt{x+2} && \text{subtract } 2x \\
5 &= 1+2\sqrt{(x-1)(x+2)} && \text{subtract } 1 \\
4 &= 2\sqrt{(x-1)(x+2)} && \text{divide by } 2 \\
2 &= \sqrt{(x-1)(x+2)} && \text{square} \\
4 &= (x-1)(x+2) && \text{FOIL} \\
4 &= x^2+x-2 && \text{reduce one seide to zero} \\
0 &= x^2+x-6 && \text{factor} \\
0 &= (x+3)(x-2) \\
x_1 &= -3 \quad \text{and} \quad x_2 = 2
\end{aligned}$$

We check: If $x=-3$, then

$$\text{LHS} = \sqrt{2(-3)+5}-\sqrt{(-3)-1} = \sqrt{-1}-\sqrt{-4} = \text{undefined}$$

Since the left hand side is undefined, $x=-3$ is NOT a solution.

If $x=2$, then

$$\begin{aligned}
\text{LHS} &= \sqrt{2(2)+5}-\sqrt{2-1} = \sqrt{9}-\sqrt{1} = 3-1 = 2 \\
\text{RHS} &= \sqrt{2+2} = \sqrt{4} = 2 \\
\text{RHS} &= \text{LHS}
\end{aligned}$$

Thus $x=2$ is the only solution.

10. $\sqrt[3]{x^3+26} = x+2$ (9.6 Example 4)

Solution:

$$\begin{aligned}
\left(\sqrt[3]{x^3+26}\right)^3 &= (x+2)^3 \\
x^3+26 &= (x+2)^3 \\
x^3+26 &= x^3+6x^2+12x+8 \\
0 &= x^3+6x^2+12x+8-x^3-26 \\
0 &= 6x^2+12x-18 \\
0 &= 6\left(x^2+2x-3\right) \\
0 &= 6(x+3)(x-1) \\
x &= -3 \quad \text{or} \quad x=1
\end{aligned}$$

We check both answers: if $x = -3$, then

$$\begin{aligned} \text{LHS} &= \sqrt[3]{(-3)^3 + 26} = \sqrt[3]{-27 + 26} = \sqrt[3]{-1} = -1 \\ \text{RHS} &= (-3) + 2 = -1 \end{aligned}$$

thus -3 does work. If $x = 1$, then

$$\begin{aligned} \text{LHS} &= \sqrt[3]{1^3 + 26} = \sqrt[3]{1 + 26} = \sqrt[3]{27} = 3 \\ \text{RHS} &= 1 + 2 = 3 \end{aligned}$$

The solution set is $\{-3, 1\}$.

11. $\sqrt{3x+1} - \sqrt{x-4} = 3$

Solution:

$$\begin{aligned} \sqrt{3x+1} - \sqrt{x-4} &= 3 && \text{add } \sqrt{x-4} \text{ to both sides} \\ \sqrt{3x+1} &= 3 + \sqrt{x-4} && \text{square both sides} \\ 3x+1 &= \left(3 + \sqrt{x-4}\right)^2 \\ 3x+1 &= \left(3 + \sqrt{x-4}\right)\left(3 + \sqrt{x-4}\right) \\ 3x+1 &= 9 + 3\sqrt{x-4} + 3\sqrt{x-4} + x - 4 \\ 3x+1 &= x + 5 + 6\sqrt{x-4} && \text{subtract } x \\ 2x+1 &= 5 + 6\sqrt{x-4} && \text{subtract } 5 \\ 2x-4 &= 6\sqrt{x-4} \\ 2(x-2) &= 6\sqrt{x-4} && \text{divide by } 2 \\ x-2 &= 3\sqrt{x-4} && \text{square both sides} \\ (x-2)^2 &= 9(x-4) && \text{FOIL, distribute} \\ x^2 - 4x + 4 &= 9x - 36 && \text{reduce one side to zero} \\ x^2 - 13x + 40 &= 0 && \text{factor} \\ (x-5)(x-8) &= 0 \\ x_1 &= 5 \quad \text{and} \quad x_2 = 8 \end{aligned}$$

We check: if $x = 5$, then

$$\begin{aligned} \text{LHS} &= \sqrt{3(5)+1} - \sqrt{5-4} = \sqrt{16} - \sqrt{1} = 4 - 1 = 3 \\ \text{RHS} &= 3 \end{aligned}$$

If $x = 8$, then

$$\begin{aligned} \text{LHS} &= \sqrt{3(8)+1} - \sqrt{8-4} = \sqrt{25} - \sqrt{4} = 5 - 2 = 3 \\ \text{RHS} &= 3 \end{aligned}$$

Solution is: $8, 5$

12. $\sqrt{x+10}+10=x$ (9.6 Example 3)

$$\begin{aligned}
\sqrt{x+10}+10 &= x && \text{subtract } 10\\
\sqrt{x+10} &= x-10 && \text{square}\\
x+10 &= (x-10)^2\\
x+10 &= x^2-20x+100 && \text{reduce one side to zero}\\
0 &= x^2-21x+90 && \text{factor}\\
0 &= (x-6)(x-15) && \text{apply the special zero property}\\
x_1 &= 6 \quad \text{and} \quad x_2 = 15
\end{aligned}$$

We check: if $x=6$, then

$$\begin{aligned}
\text{LHS} &= \sqrt{6+10}+10=\sqrt{16}+10=4+10=14\\
\text{RHS} &= 6\\
\text{LHS} &\neq \text{RHS}
\end{aligned}$$

If $x=15$, then

$$\begin{aligned}
\text{LHS} &= \sqrt{15+10}+10=\sqrt{25}+10=5+10=15\\
\text{RHS} &= 15
\end{aligned}$$

since $x=6$ doesn't work, the only solution is 15. In set notation: the solution set is $\{15\}$.

13. $\sqrt[3]{x^3+208}=x+4$ (9.6 Example 4)

Solution:

$$\begin{aligned}
\sqrt[3]{x^3+208} &= x+4 && \text{raise both sides to the third power}\\
\left(\sqrt[3]{x^3+208}\right)^3 &= (x+4)^3\\
x^3+208 &= x^3+12x^2+48x+64 && \text{subtract } x^3\\
208 &= 12x^2+48x+64 && \text{reduce one side to zero}\\
0 &= 12x^2+48x-144 && \text{factor out 12}\\
0 &= 12\left(x^2+4x-12\right) && \text{factor}\\
0 &= 12(x+6)(x-2)\\
x_1 &= -6 \quad \text{and} \quad x=2
\end{aligned}$$

Solution: -6 and 2. We check. They both work.

14. $\sqrt{x-1}+\sqrt{x-4}=\sqrt{4x-11}$

Solution:

$$\begin{aligned}
\sqrt{x-1}+\sqrt{x-4} &= \sqrt{4x-11} && \text{square b} \\
\left(\sqrt{x-1}+\sqrt{x-4}\right)^2 &= \left(\sqrt{4x-11}\right)^2 && \\
\left(\sqrt{x-1}+\sqrt{x-4}\right)\left(\sqrt{x-1}+\sqrt{x-4}\right) &= 4x-11 && \text{FOIL} \\
\underbrace{\sqrt{x-1}\sqrt{x-1}}_{\text{F}}+\underbrace{\sqrt{x-1}\sqrt{x-4}}_{\text{O}}+\underbrace{\sqrt{x-4}\sqrt{x-1}}_{\text{I}}+\underbrace{\sqrt{x-4}\sqrt{x-4}}_{\text{L}} &= 4x-11 && \\
x-1+2\sqrt{x-1}\sqrt{x-4}+x-4 &= 4x-11 && \text{combine like t} \\
2x-5+2\sqrt{x-1}\sqrt{x-4} &= 4x-11 && \text{subtract } 2x \\
-5+2\sqrt{x-1}\sqrt{x-4} &= 2x-11 && \text{add } 5 \\
2\sqrt{x-1}\sqrt{x-4} &= 2x-6 && \\
2\sqrt{x-1}\sqrt{x-4} &= 2(x-3) && \text{divide by } 2 \\
\sqrt{(x-1)(x-4)} &= x-3 && \text{square} \\
(x-1)(x-4) &= (x-3)^2 && \text{FOIL} \\
x^2-5x+4 &= x^2-6x+9 && \text{subtract } x^2 \\
-5x+4 &= -6x+9 && \text{add } 6x \\
x+4 &= 9 && \text{subtract } 4 \\
x &= 5 &&
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \sqrt{5-1}+\sqrt{5-4}=\sqrt{4}+\sqrt{1}=2+1=3 \\
\text{RHS} &= \sqrt{4x-11}=\sqrt{4(5)-11}=\sqrt{9}=3
\end{aligned}$$

And so the solution set is $\{5\}$.

15. $\sqrt{4x+6}=\sqrt{x+1}-\sqrt{x+5}$

Solution:

$$\begin{aligned}
\sqrt{4x+6} &= \sqrt{x+1}-\sqrt{x+5} && \text{square}\\
\left(\sqrt{4x+6}\right)^2 &= \left(\sqrt{x+1}-\sqrt{x+5}\right)^2 &&\\
4x+6 &= \underbrace{\sqrt{x+1}\sqrt{x+1}}_{\text{F}}-\underbrace{\sqrt{x+1}\sqrt{x+5}}_{\text{O}}-\underbrace{\sqrt{x+5}\sqrt{x+1}}_{\text{I}}+\underbrace{\sqrt{x+5}\sqrt{x+5}}_{\text{L}} &&\\
4x+6 &= x+1-2\sqrt{(x+1)(x+5)}+x+5 &&\\
4x+6 &= 2x+6-2\sqrt{(x+1)(x+5)} && \text{subtract } 2x\\
2x+6 &= 6-2\sqrt{(x+1)(x+5)} && \text{subtract } 6\\
2x &= -2\sqrt{(x+1)(x+5)} && \text{divide by } 2\\
x &= -\sqrt{(x+1)(x+5)} && \text{square}\\
x^2 &= \left(-\sqrt{(x+1)(x+5)}\right)^2 &&\\
x^2 &= (x+1)(x+5) && \text{FOIL right hand side}\\
x^2 &= x^2+6x+5 && \text{subtract } x^2\\
0 &= 6x+5 && \text{subtract } 5\\
-5 &= 6x && \text{divide by } 6\\
-\frac{5}{6} &= x &&
\end{aligned}$$

We check: if $x=-\frac{5}{6}$, then

$$\begin{aligned}
\text{LHS} &= \sqrt{4\left(-\frac{5}{6}\right)+6}=\sqrt{-\frac{10}{3}+6}=\sqrt{-\frac{10}{3}+\frac{18}{3}}=\sqrt{\frac{8}{3}}\\
\text{RHS} &= \sqrt{-\frac{5}{6}+1}-\sqrt{-\frac{5}{6}+5}=\sqrt{\frac{1}{6}}-\sqrt{\left(-\frac{5}{6}\right)+\frac{30}{6}}=\sqrt{\frac{1}{6}}-\sqrt{\frac{25}{6}}=\frac{\sqrt{1}}{\sqrt{6}}-\frac{\sqrt{25}}{\sqrt{6}}\\
&= \frac{1}{\sqrt{6}}-\frac{5}{\sqrt{6}}=\frac{1-5}{\sqrt{6}}=-\frac{4}{\sqrt{6}}
\end{aligned}$$

Since the left hand side is positive, and the right hand side is negative, these two numbers can not be equal. There is no solution.